THE CARTOON GUIDE TO STATISTICS

LARRY GONICK

Author of *The Cartoon History of the Universe*

& WOOLLCOTT SMITH

Also by Larry Gonick

The Cartoon History of the Universe
The Cartoon Guide to Physics (with Art Huffman)
The Cartoon Guide to the Computer
The Cartoon Guide to Genetics (with Mark Wheelis)
The Cartoon History of the United States
The Cartoon Guide to (Non) Communication

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WILLIAM FAIRLEY'S AND LEAH SMITH'S COMMENTS IMPROVED EARLIER DRAFTS OF THIS BOOK

DONNA OKINO PROVIDED INVALUABLE ASSISTANCE AND ADVICE IN PRODUCING THE CARTOON PAGES SHE SAYS THAT CREATING A CARTOON GUIDE IS HARDER THAN RUNNING A MARATHON, AND SHE SHOULD KNOW; SHE'S DONE BOTH.

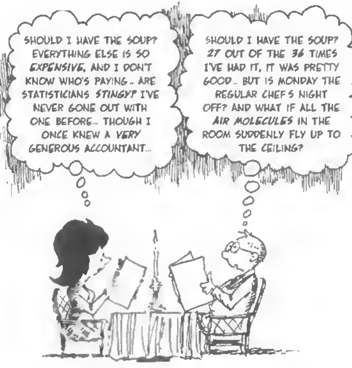
THE ALTSYS CORPORATION CREATED FONTOGRAPHER, THE WONDERFUL SOFTWARE THAT ALLOWED US TO SIMULATE HAND-LETTERED TEXT AND FORMULAS ON THE MACINTOSH.

AND, SINCE EDUCATION IS ALWAYS A TWO-WAY STREET, A TIP OF THE HAT TO SMITH'S LONG-SUFFERING TEMPLE UNIVERSITY STUDENTS AND ESPECIALLY THE FALL '92 STUDY GROUP ORGANIZED BY ADRIANA TORRES. THE FUTURE IS THEIRS

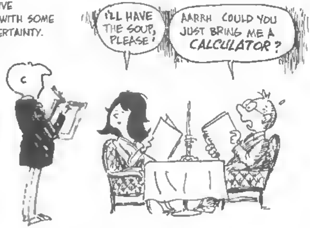
♦Chapter 1♦

WHAT IS STATISTICS?

WE MUDDLE THROUGH LIFE MAKING CHOICES BASED ON *INCOMPLETE INFORMATION*...



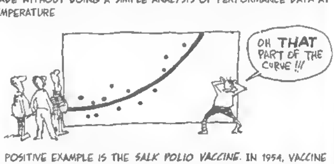
MOST OF US LIVE COMFORTABLY WITH SOME LEVEL OF UNCERTAINTY.



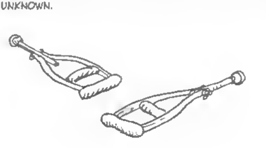
WHAT MAKES STATISTICS UNIQUE IS ITS ABILITY TO *QUANTIFY* UNCERTAINTY, TO MAKE IT PRECISE. THIS ALLOWS STATISTICIANS TO MAKE *CATEGORICAL STATEMENTS*, WITH COMPLETE ASSURANCE—ABOUT THEIR LEVEL OF UNCERTAINTY!

THIS IS NOT JUST A MATTER OR ORDERING SOUP! STATISTICS ALSO INVOLVES MATTERS OF *LIFE AND DEATH...*

FOR EXAMPLE, IN 1986, THE SPACE SHUTTLE *CHALLENGER* EXPLODED, KILLING SEVEN ASTRONAUTS. THE DECISION TO LAUNCH IN 29-DEGREE WEATHER HAD BEEN MADE WITHOUT DOING A SIMPLE ANALYSIS OF PERFORMANCE DATA AT LOW TEMPERATURE



A MORE POSITIVE EXAMPLE IS THE *SALK POLIO VACCINE*. IN 1954, VACCINE TRIALS WERE PERFORMED ON SOME *400,000* CHILDREN, WITH STRICT CONTROLS TO ELIMINATE BIASED RESULTS. GOOD STATISTICAL ANALYSIS OF THE RESULTS FIRMLY ESTABLISHED THE VACCINE'S EFFECTIVENESS, AND TODAY POLIO IS ALMOST UNKNOWN.



TO ACCOMPLISH THEIR FEATS OF MATHEMATICAL LEGERDEMAIN, STATISTICIANS RELY ON THREE RELATED DISCIPLINES

Data analysis

THE GATHERING, DISPLAY, AND SUMMARY OF DATA.

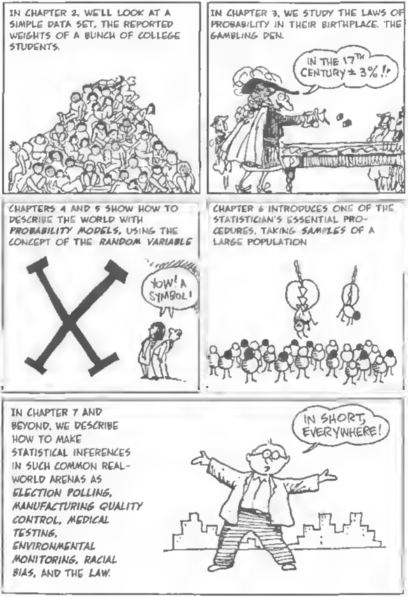
Probability

THE LAWS OF CHANCE, IN AND OUT OF THE CASINO;

Statistical inference

THE SCIENCE OF DRAWING STATISTICAL CONCLUSIONS FROM SPECIFIC DATA USING A KNOWLEDGE OF PROBABILITY

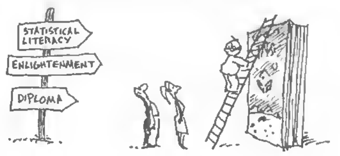
IN THIS BOOK, WE'LL LOOK AT ALL THREE, AS APPLIED TO A WIDE VARIETY OF SITUATIONS WHERE STATISTICS PLAYS A CRUCIAL ROLE IN THE MODERN WORLD.


IN CHAPTER 2, WE'LL LOOK AT A SIMPLE DATA SET, THE REPORTED WEIGHTS OF A BUNCH OF COLLEGE STUDENTS.
IN CHAPTER 3, WE STUDY THE LAWS OF PROBABILITY IN THEIR BIRTHPLACE, THE GAMBLING DEN.
IN THE 17TH CENTURY ± 3%!!
CHAPTERS 4 AND 5 SHOW HOW TO DESCRIBE THE WORLD WITH PROBABILITY MODELS, USING THE CONCEPT OF THE RANDOM VARIABLE
YOW! A SYMBOL!
CHAPTER 6 INTRODUCES ONE OF THE STATISTICIAN'S ESSENTIAL PROCEDURES, TAKING SAMPLES OF A LARGE POPULATION
IN CHAPTER 7 AND BEYOND, WE DESCRIBE HOW TO MAKE STATISTICAL INFERENCES IN SUCH COMMON REAL-WORLD ARENAS AS ELECTION POLLING, MANUFACTURING QUALITY CONTROL, MEDICAL TESTING, ENVIRONMENTAL MONITORING, RACIAL BIAS, AND THE LAW.
IN SHORT, EVERYWHERE!

FINALLY, IN DISCUSSING STATISTICS, IT'S HARD TO AVOID MENTIONING ONE OTHER THING: THE WIDESPREAD *MISTRUST* OF STATISTICS IN THE WORLD TODAY. EVERYONE KNOWS ABOUT "LYING WITH STATISTICS," WHILE GOOD STATISTICAL ANALYSIS IS NEARLY IMPOSSIBLE TO FIND IN DAILY LIFE. WHAT'S ONE TO DO?

OUR HUMBLE OPINION IS THAT *LEARNING A LITTLE MORE ABOUT THE SUBJECT* MIGHT NOT BE SUCH A BAD IDEA. AND THAT'S WHY WE WROTE THIS BOOK!

IN WHAT FOLLOWS, WE TRY TO PRESENT THE ELEMENTS OF STATISTICS AS GRAPHICALLY AND INTUITIVELY AS POSSIBLE. ALL YOU NEED TO GET THROUGH IT IS A LITTLE *PATIENCE*, SOME *THOUGHT*, AND A CERTAIN TOLERANCE FOR *ALGEBRA*—OR, IF NOT THAT, THEN MAYBE A *COURSE REQUIREMENT!!*



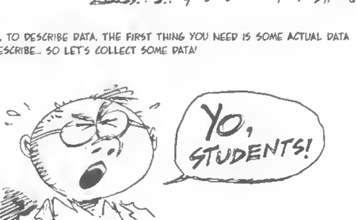
♦CHAPTER 2♦

DATA DESCRIPTION

DATA ARE THE STATISTICIAN'S RAW MATERIAL, THE NUMBERS WE USE TO INTERPRET REALITY. ALL STATISTICAL PROBLEMS INVOLVE EITHER THE *COLLECTION*, DESCRIPTION, AND ANALYSIS OF DATA, OR *THINKING* ABOUT THE COLLECTION, DESCRIPTION, AND ANALYSIS OF DATA.

THIS CHAPTER CONCENTRATES ON DATA ***DESCRIPTION***. HOW CAN WE REPRESENT DATA IN USEFUL WAYS? HOW CAN WE SEE UNDERLYING PATTERNS IN A HEAP OF NAKED NUMBERS? HOW CAN WE SUMMARIZE THE DATA'S BASIC SHAPE?

WELL, TO DESCRIBE DATA, THE FIRST THING YOU NEED IS SOME ACTUAL DATA TO DESCRIBE... SO LET'S COLLECT SOME DATA!



HERE IS SOME REAL DATA: AS PART OF A CLASSROOM EXPERIMENT, 92 PENN STATE STUDENTS REPORTED THEIR WEIGHT, WITH THESE RESULTS:

MALES

140 145 160 190 155 165 150 190 195 138 160 155 153 145 170 175 175 170 180 135 170 157 130 185 190 155 170 155 215 150 145 155 155 150 155 150 180 160 135 160 130 155 150 148 155 150 140 180 190 145 150 164 140 142 136 123 155

FEMALES

140 120 130 138 121 125 116 145 150 112 125 130 120 130 131 120 118 125 135 125 118 122 115 102 115 150 110 116 108 95 125 133 110 150 108

GETTING RIGHT DOWN TO BUSINESS, WE DRAW A *DOT PLOT:* ONE DOT PER STUDENT GOES OVER EACH STUDENT'S REPORTED WEIGHT.

YOU MAY SEE A *PROBLEM* HERE: THE CLUMPS AT *150* AND *155* POUNDS. THE STUDENTS TENDED TO REPORT THEIR WEIGHT IN *FIVE-POUND INCREMENTS*. IN REAL-LIFE SITUATIONS LIKE THIS ONE, SUCH ROUNDING OFF CAN OBSCURE GENERAL PATTERNS IN DATA... BUT FOR NOW, WE'LL JUST WORK AROUND IT

WE CAN SUMMARIZE THE DATA WITH A *FREQUENCY TABLE.* DIVIDE THE NUMBER LINE INTO INTERVALS AND COUNT THE NUMBER OF STUDENT WEIGHTS WITHIN EACH INTERVAL. THE *FREQUENCY* IS THE COUNT IN ANY GIVEN INTERVAL. THE *RELATIVE FREQUENCY* IS THE PROPORTION OF WEIGHTS IN EACH INTERVAL, I.E., IT'S THE FREQUENCY DIVIDED BY THE TOTAL NUMBER OF STUDENTS.

CLASS INTERVAL	MIDPOINT	FREQUENCY	RELATIVE FREQUENCY
87.5-102.4	95	2	.022
102.5-117.5	110	9	.098
117.5-132.4	125	19	.206
132.5-147.4	140	17	.185
147.5-162.4	155	27	.293
162.5-177.4	170	8	.087
177.5-192.4	185	8	.087
192.5-207.5	200	1	.011
207.5-222.4	215	1	.011
TOTAL		92	1.000

NOTE: WE KEPT THE INTERVAL BOUNDARIES AWAY FROM THOSE TROUBLESOME 5-POUND MULTIPLES. THIS GETS AROUND THE STUDENTS' REPORTING BIAS

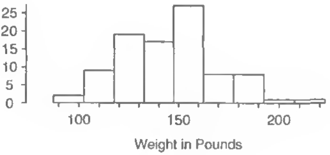
GUIDELINES FOR FORMING THE CLASS INTERVALS.

1) USE INTERVALS OF EQUAL LENGTH WITH MIDPOINTS AT CONVENIENT ROUND NUMBERS.

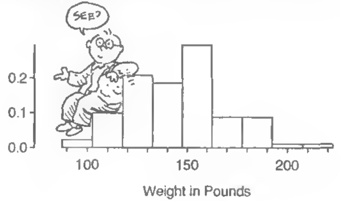
2) FOR A SMALL DATA SET, USE A SMALL NUMBER OF INTERVALS.

3) FOR A LARGE DATA SET, USE MORE INTERVALS!

IN THE FREQUENCY TABLE, WE ARE SHOWING HOW MANY DATA POINTS ARE "AROUND" EACH VALUE. WE CAN GRAPH THIS INFORMATION, TOO. THE RESULTING BAR GRAPH IS CALLED A *HISTOGRAM*. EACH BAR COVERS AN INTERVAL AND IS CENTERED AT THE MIDPOINT. THE BAR'S HEIGHT IS THE NUMBER OF DATA POINTS IN THE INTERVAL.



WE CAN ALSO DRAW A *RELATIVE FREQUENCY HISTOGRAM*, PLOTTING THE RELATIVE FREQUENCY AGAINST THE WEIGHT. IT LOOKS EXACTLY THE SAME, EXCEPT FOR THE VERTICAL SCALE.



THE STATISTICIAN *JOHN TUKEY* INVENTED A QUICK WAY TO SUMMARIZE DATA AND STILL KEEP THE INDIVIDUAL DATA POINTS. IT'S CALLED THE ***STEM-AND-LEAF*** DIAGRAM.

FOR THE WEIGHT DATA, THE STEM IS A COLUMN OF NUMBERS, CONSISTING OF THE WEIGHT DATA COUNTED BY TENS (I.E. WE LEAVE OFF THE LAST DIGIT).

9
10
11
12
13
14
15
16
17
18
19
20
21

NOW ADD THE FINAL DIGIT OF EACH WEIGHT IN THE APPROPRIATE ROW

STEM	LEAVES
9	
10	
11	628
12	0155005
13	080015
14	05
15	0
16	
17	
18	
19	
20	
21	

FILLED IN, IT LOOKS LIKE THIS

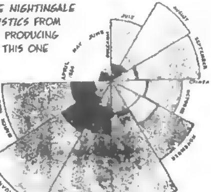
9	5
10	288
11	628855060
12	01553005525
13	8500050600153
14	05505580502
15	505370550550505050500500
16	050004
17	055000
18	0500
19	00500
20	
21	5

AND FINALLY, PUT THE "LEAVES" IN ORDER

9	5
10	288
11	002556688
12	00012355555
13	0000013555688
14	00002555558
15	00000000000355555555557
16	000045
17	000055
18	0005
19	00005
20	
21	5

ALL THOSE ZEROES AND FIVES CLEARLY SHOW THE STUDENTS' REPORTING BIAS!

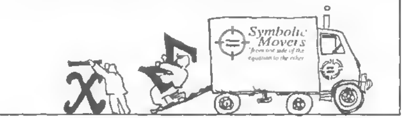
CRUSADING NURSE *FLORENCE NIGHTINGALE* COMPILED *MORTALITY STATISTICS* FROM BRITISH MILITARY HOSPITALS, PRODUCING SHOCKING HISTOGRAMS LIKE THIS ONE. THE RADIAL AXIS INDICATES DEATHS—IN HOSPITALS AS WELL AS ON THE BATTLEFIELD—OF BRITISH SOLDIERS IN THE CRIMEAN WAR



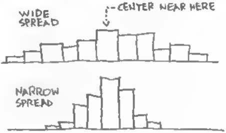
HER STATISTICAL EFFORTS LED DIRECTLY TO IMPROVED HOSPITAL CONDITIONS AND A REDUCTION IN THE DEATH RATE.

SUMMARY STATISTICS

NOW WE MOVE FROM PICTURES TO FORMULAS. OUR OBJECT IS TO GET SOME SIMPLE MEASUREMENTS OF THE CRUDEST CHARACTERISTICS OF A SET OF DATA...



ANY SET OF MEASUREMENTS HAS TWO IMPORTANT PROPERTIES: THE *CENTRAL OR TYPICAL VALUE*, AND THE *SPREAD* ABOUT THAT VALUE. YOU CAN SEE THE IDEA IN THESE HYPOTHETICAL HISTOGRAMS



WE CAN GO A LONG WAY WITH A LITTLE NOTATION. SUPPOSE WE'RE MAKING A SERIES OF OBSERVATIONS... n OF THEM, TO BE EXACT... THEN WE WRITE

$$x_1, x_2, x_3, \ldots x_n$$

AS THE VALUES WE OBSERVE. THUS, n IS THE TOTAL NUMBER OF DATA POINTS, AND x_4 (SAY) IS THE VALUE OF THE FOURTH DATA POINT.

AN *ARRAY* IS A TABLE OF DATA.

OBSERVATION	1	2	3	4	...	n
DATA VALUE	x_1	x_2	x_3	x_4	...	x_n

A SMALL SET OF $n = 5$ DATA POINTS MAKES THE BOOKKEEPING EASY. SUPPOSE, FOR EXAMPLE, WE ASK FIVE PEOPLE HOW MANY HOURS OF *TELEVISION* THEY WATCH IN A WEEK... AND GET THE FOLLOWING ARRAY:

OBSERVATION	1	2	3	4	5
DATA VALUE	5	7	3	38	7

THEN $x_1 = 5$, $x_2 = 7$, $x_3 = 3$, $x_4 = 38$, AND $x_5 = 7$.

WHAT'S THE "CENTER" OF THESE DATA? THERE ARE ACTUALLY SEVERAL DIFFERENT WAYS TO MEASURE IT. WE'LL LOOK AT JUST TWO OF THEM

THE **MEAN** (OR "AVERAGE")

THE *MEAN* OR AVERAGE VALUE IS REPRESENTED BY $\bar{x}$... IT'S OBTAINED BY ADDING ALL THE DATA AND DIVIDING BY THE NUMBER OF OBSERVATIONS:

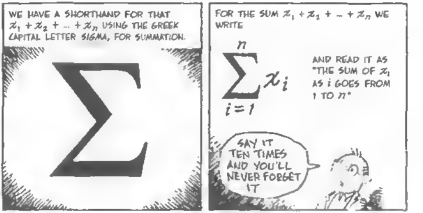
$$\bar{x} = \frac{\text{SUM OF DATA}}{n}$$

$$= \frac{x_1 + x_2 + \dots + x_n}{n}$$

FOR OUR EXAMPLE,

$$\bar{x} = \frac{5 + 7 + 3 + 38 + 7}{5} = \frac{60}{5}$$

$= 12$ HOURS



SO... TO REPEAT, THE AVERAGE, OR *MEAN*, OF A SET OF DATA x_i IS

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \quad \text{OR} \quad \sum_{i=1}^{n} \frac{x_i}{n}$$

IN THE CASE OF OUR 92 PENN STATE STUDENTS, THE MEAN WEIGHT IS

$$\sum_{i=1}^{92} \frac{x_i}{92} = \frac{13{,}354}{92}$$

$$=$$

145.15 POUNDS

THE MEDIAN

IS ANOTHER KIND OF CENTER. THE "MIDPOINT" OF THE DATA, LIKE THE "MEDIAN STRIP" IN A ROAD

TO FIND THE MEDIAN VALUE OF A DATA SET, WE ARRANGE THE DATA IN ORDER FROM SMALLEST TO LARGEST THE MEDIAN IS THE VALUE IN THE MIDDLE

3 5 7 7 38

THE MEDIAN

IF THE NUMBER OF POINTS IS *EVEN*—IN WHICH CASE THERE IS NO MIDDLE, WE AVERAGE THE TWO VALUES AROUND THE MIDDLE... SO IF THE DATA ARE

3 5 7 7

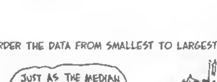
MIDDLE SPACE

WE AVERAGE 5 AND 7 TO GET $\frac{5+7}{2} = 6$

THIS GIVES US A GENERAL RULE: ORDER THE DATA FROM SMALLEST TO LARGEST.

IF THE NUMBER OF DATA POINTS IS *ODD*, THE MEDIAN IS THE MIDDLE DATA POINT.

IF THE NUMBER OF POINTS IS *EVEN*, THE MEDIAN IS THE AVERAGE OF THE TWO DATA POINTS NEAREST THE MIDDLE.



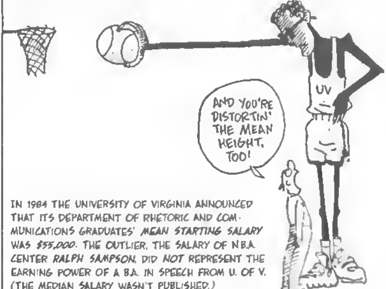
FOR THE $n = 92$ STUDENT WEIGHTS, WE CAN FIND THE MEDIAN FROM THE ORDERED STEM-AND-LEAF DIAGRAM JUST COUNT TO THE 46TH OBSERVATION. THE MEDIAN IS

$$\frac{x_{46} + x_{47}}{2} = \frac{145 + 145}{2}$$

$$= \mathbf{145} \text{ POUNDS}$$

9	5
10	288
11	002556688
12	00012355555
13	0000013555688
14	0000255555 8
15	0000000000355555555557
16	000045
17	000055
18	0005
19	00005
20	
21	5

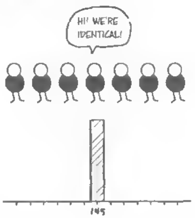
WHY MORE THAN ONE MEASURE OF THE CENTER? EACH HAS ADVANTAGES. FOR EXAMPLE, THE *MEDIAN* IS NOT SENSITIVE TO *OUTLIERS*, OR EXTREME VALUES NOT TYPICAL OF THE REST OF THE DATA SUPPOSE IN OUR SMALL TV-WATCHING GROUP, ONE PERSON WATCHES **200** HOURS PER WEEK. THEN OUR DATA ARE 3, 5, 7, 7, 200 THE MEDIAN, 7, IS UNCHANGED, BUT THE MEAN IS NOW $\bar{x} = 45.8$!



IN 1984 THE UNIVERSITY OF VIRGINIA ANNOUNCED THAT ITS DEPARTMENT OF RHETORIC AND COMMUNICATIONS GRADUATES' *MEAN STARTING SALARY* WAS *\$55,000*. THE OUTLIER, THE SALARY OF N.B.A. CENTER *RALPH SAMPSON*, DID *NOT* REPRESENT THE EARNING POWER OF A B.A. IN SPEECH FROM U. OF V. (THE MEDIAN SALARY WASN'T PUBLISHED.)

MEASURES OF SPREAD

BESIDES KNOWING THE CENTRAL POINT OF A DATA SET, WE'D ALSO LIKE TO DESCRIBE THE DATA'S *SPREAD*, OR HOW FAR FROM THE CENTER THE DATA TEND TO RANGE. FOR INSTANCE, IF THE STUDENTS ALL WEIGHED *EXACTLY* 145 POUNDS, THERE WOULD BE NO SPREAD AT ALL. NUMERICALLY, THE SPREAD WOULD BE *ZERO*, AND THE HISTOGRAM WOULD BE SKINNY.



BUT IF MANY OF THE STUDENTS WERE VERY LIGHT AND/OR VERY HEAVY, OBVIOUSLY WE'D SEE SOME SPREAD—SAY, IF THE ***FOOTBALL TEAM*** WAS PART OF THE SAMPLE...

THE HISTOGRAM WOULD BE WIDER, SOMETHING LIKE THIS.

AGAIN, THERE'S MORE THAN ONE WAY TO MEASURE A SPREAD. ONE WAY IS

INTERQUARTILE RANGE

THE IDEA IS TO DIVIDE THE DATA INTO FOUR EQUAL GROUPS AND SEE HOW FAR APART THE EXTREME GROUPS ARE

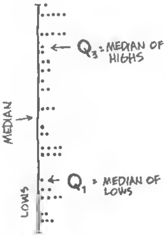
HERE'S THE RECIPE

1) PUT THE DATA IN NUMERICAL ORDER

2) DIVIDE THE DATA INTO TWO EQUAL HIGH AND LOW GROUPS AT THE MEDIAN (IF THE MEDIAN IS A DATA POINT, INCLUDE IT IN BOTH THE HIGH AND LOW GROUPS.)

3) FIND THE MEDIAN OF THE LOW GROUP THIS IS CALLED THE FIRST QUARTILE, OR Q_1.

4) THE MEDIAN OF THE HIGH GROUP IS THE THIRD QUARTILE, OR Q_3



NOW THE INTERQUARTILE RANGE (IQR) IS THE DISTANCE (OR DIFFERENCE) BETWEEN THEM.

$$IQR = Q_3 - Q_1$$

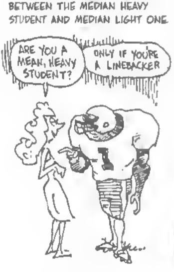
HERE'S THE WEIGHT DATA WITH THE MIDPOINTS OF THE HIGH AND LOW GROUPS EMPHASIZED:

9 · 5
10 . 288
11 . 00255668
12 . 00012355555
13 · 0000013555688
14 . 00002555558
15 : 0000000000355555555557
16 : 000045
17 . 000055
18 . 0005
19 : 00005
20:
21 5

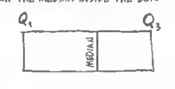
AND WE SEE THAT

$$IQR = 156 - 125$$
$$= 31 \text{ POUNDS}$$

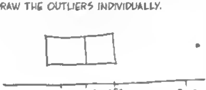
AGAIN, THIS IS THE DIFFERENCE BETWEEN THE MEDIAN HEAVY STUDENT AND MEDIAN LIGHT ONE.



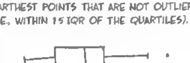
JOHN TUKEY INVENTED ANOTHER KIND OF DISPLAY TO SHOW OFF THE IQR, CALLED A ***BOX AND WHISKERS*** PLOT. THE BOX'S ENDS ARE THE QUARTILES Q_1 AND Q_3. WE DRAW THE MEDIAN INSIDE THE BOX.

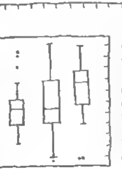


IF A POINT IS MORE THAN 1.5 IQR FROM AN END OF THE BOX, IT'S AN *OUTLIER*. DRAW THE OUTLIERS INDIVIDUALLY.



FINALLY, EXTEND "WHISKERS" OUT TO THE FARTHEST POINTS THAT ARE NOT OUTLIERS (I.E., WITHIN 1.5 IQR OF THE QUARTILES).





BOX-AND-WHISKERS PLOTS ARE ESPECIALLY GOOD FOR SHOWING OFF DIFFERENCES BETWEEN GROUPS.

THE STANDARD MEASURE OF SPREAD IS THE

STANDARD DEVIATION

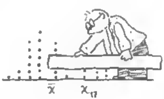
UNLIKE THE IQR, WHICH IS BASED ON MEDIANS, THE STANDARD DEVIATION MEASURES THE SPREAD FROM THE *MEAN*. YOU CAN THINK OF IT, ROUGHLY SPEAKING, AS THE AVERAGE DISTANCE OF THE DATA FROM THE MEAN $\bar{x}$...

EXCEPT THAT WE USE THE *SQUARES* OF THE DISTANCES INSTEAD. THAT IS, IF THE SQUARED DISTANCE OF POINT x_i TO $\bar{x}$ IS $(x_i - \bar{x})^2$, THEN

$$\text{AVERAGE SQUARED DISTANCE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

FOR TECHNICAL REASONS, WE USE $n-1$ IN THE DENOMINATOR RATHER THAN n, AND DEFINE THE *SAMPLE VARIANCE* s^2 AS

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

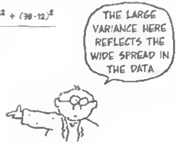


FOR THE DATA SET $\{3 \quad 5 \quad 7 \quad 7 \quad 38\}$, WITH $\bar{x} = 12$ AND $n = 5$ WE CALCULATE THE VARIANCE:

$$s^2 = \frac{(3-12)^2 + (5-12)^2 + (7-12)^2 + (7-12)^2 + (38-12)^2}{(5-1)}$$

$$= \frac{81 + 49 + 25 + 25 + 676}{4}$$

$$= 214$$



BUT A SPREAD MEASURE SHOULD HAVE THE SAME *UNITS* AS THE ORIGINAL DATA. IN THE EXAMPLE OF WEIGHTS, THE VARIANCE s^2 IS MEASURED IN POUNDS *SQUARED*... OOOPS!

THE OBVIOUS THING TO DO IS TO TAKE THE SQUARE ROOT, AND SO WE DO... TO DEFINE:

STANDARD DEVIATION

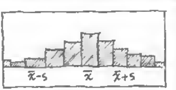
$$s = \sqrt{s^2} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

WHICH, FOR OUR SIMPLE DATA SET, IS

$$s = \sqrt{214} = 14.63$$

EVEN FOR SMALL DATA SETS, THE ARITHMETIC CAN BE TEDIOUS! SO NOWADAYS, WE JUST HIT THE s BUTTON ON THE HAND CALCULATOR, OR CONSULT THE DATA REPORT GENERATED BY A COMPUTER SOFTWARE PACKAGE.

Properties of $\overline{X}$ and S

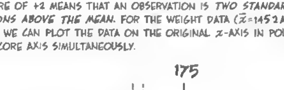


THE MEAN AND STANDARD DEVIATION ARE VERY GOOD FOR SUMMARIZING THE PROPERTIES OF FAIRLY ***SYMMETRICAL HISTOGRAMS WITHOUT OUTLIERS***—I.E., HISTOGRAMS SHAPED LIKE MOUNDS

IT'S OFTEN USEFUL TO KNOW HOW MANY STANDARD DEVIATIONS A DATA POINT IS FROM THE MEAN. WE DEFINE ***Z-SCORES***, OR STANDARDIZED SCORES, AS DISTANCE FROM $\bar{x}$ ***PER STANDARD DEVIATION***.

$$z_i = \frac{x_i - \bar{x}}{s} \quad \text{FOR EACH } i.$$

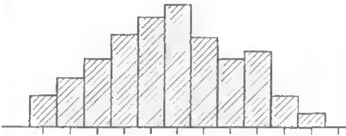
A Z-SCORE OF +2 MEANS THAT AN OBSERVATION IS ***TWO STANDARD DEVIATIONS ABOVE THE MEAN***. FOR THE WEIGHT DATA ($\bar{x}$=145.2 AND s=23.7), WE CAN PLOT THE DATA ON THE ORIGINAL x-AXIS IN POUNDS AND THE Z-SCORE AXIS SIMULTANEOUSLY.



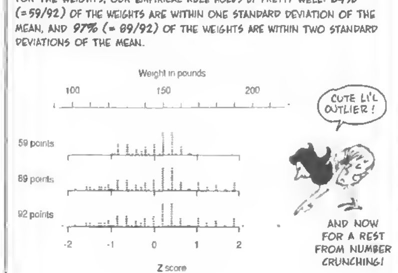
A STUDENT WEIGHING 175 POUNDS HAS A Z-SCORE OF $\frac{175-145.2}{23.7} = \mathbf{1.26}$

an EMPIRICAL RULE:

FOR NEARLY SYMMETRIC MOUND-SHAPED DATA SETS, APPROXIMATELY *68%* OF THE DATA IS WITHIN *ONE* STANDARD DEVIATION OF THE MEAN AND *95%* OF THE DATA IS WITHIN *TWO* STANDARD DEVIATIONS OF THE MEAN



FOR THE WEIGHTS, OUR EMPIRICAL RULE HOLDS UP PRETTY WELL: *64%* (=59/92) OF THE WEIGHTS ARE WITHIN ONE STANDARD DEVIATION OF THE MEAN, AND *97%* (= 89/92) OF THE WEIGHTS ARE WITHIN TWO STANDARD DEVIATIONS OF THE MEAN.



WE'VE COME A LONG WAY IN THIS CHAPTER! STARTING WITH A UNORGANIZED PILE OF NUMBERS, WE HAVE:

1) FOUND SEVERAL DIFFERENT WAYS TO DISPLAY THEM

2) LOOKED AT TWO DIFFERENT CONCEPTS OF THE CENTER OF DATA, THE MEDIAN AND THE MEAN

3) MEASURED THE SPREAD OF THE DATA AROUND THE CENTER IN TWO DIFFERENT WAYS

4) ENCOUNTERED MOUND-SHAPED HISTOGRAMS AND Z, A VARIABLE THAT INDICATES HOW MANY STANDARD DEVIATIONS YOU ARE FROM THE MEAN

NOW, IN ORDER TO PROBE THE BEHAVIOR OF DATA MORE DEEPLY, WE'RE GOING TO MAKE A LITTLE DETOUR INTO THE REALM OF *RANDOMNESS*... A LAND WHERE THINGS *ALWAYS* WORK OUT IN THE LONG RUN, AND WHERE THE ONLY LAW IS THE LAW OF THE *GAMBLING CASINO*...

♦Chapter 3♦

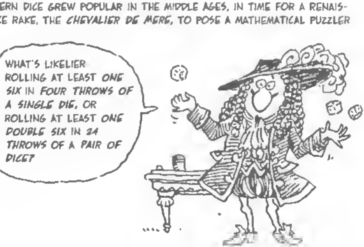
PROBABILITY

NOTHING IN LIFE IS CERTAIN IN EVERYTHING WE DO, WE GAUGE THE CHANCES OF SUCCESSFUL OUTCOMES, FROM BUSINESS TO MEDICINE TO THE WEATHER. BUT FOR MOST OF HUMAN HISTORY, *PROBABILITY*, THE FORMAL STUDY OF THE LAWS OF CHANCE, WAS USED FOR ONLY ONE THING: *GAMBLING*.

NOBODY KNOWS WHEN GAMBLING BEGAN. IT GOES BACK AT LEAST AS FAR AS *ANCIENT EGYPT*, WHERE SPORTING MEN AND WOMEN USED FOUR-SIDED *"ASTRAGALI"* MADE FROM ANIMAL HEELBONES

THE ROMAN EMPEROR *CLAUDIUS* (10 BCE-54 CE) WROTE THE FIRST KNOWN TREATISE ON GAMBLING. UNFORTUNATELY, THIS BOOK, "HOW TO WIN AT DICE," WAS LOST.

MODERN DICE GREW POPULAR IN THE MIDDLE AGES, IN TIME FOR A RENAISSANCE RAKE, THE *CHEVALIER DE MERE*, TO POSE A MATHEMATICAL PUZZLER



THE CHEVALIER REASONED THAT THE AVERAGE NUMBER OF SUCCESSFUL ROLLS WAS THE SAME FOR BOTH GAMBLES:

CHANCE OF ONE SIX $= \frac{1}{6}$

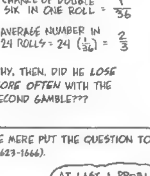
AVERAGE NUMBER IN FOUR ROLLS $= 4\left(\frac{1}{6}\right) = \frac{2}{3}$

CHANCE OF DOUBLE SIX IN ONE ROLL $= \frac{1}{36}$

AVERAGE NUMBER IN 24 ROLLS $= 24\left(\frac{1}{36}\right) = \frac{2}{3}$

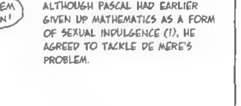
WHY, THEN, DID HE *LOSE MORE OFTEN* WITH THE SECOND GAMBLE???

DE MERE PUT THE QUESTION TO HIS FRIEND, THE GENIUS *BLAISE PASCAL* (1623-1666).



ALTHOUGH PASCAL HAD EARLIER GIVEN UP MATHEMATICS AS A FORM OF SEXUAL INDULGENCE (!), HE AGREED TO TACKLE DE MERE'S PROBLEM.

PASCAL WROTE HIS FELLOW GENIUS *PIERRE DE FERMAT*, AND WITHIN A FEW LETTERS, THE TWO HAD WORKED OUT THE THEORY OF PROBABILITY IN ITS MODERN FORM—EXCEPT, OF COURSE, FOR THE CARTOONS



BASIC DEFINITIONS

AS OUR GAMBLER PLAYS A GAME, WE PLAY SCIENTIST, OBSERVING THE OUTCOME:

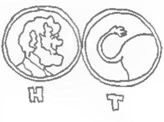
A **random experiment** IS THE PROCESS OF OBSERVING THE OUTCOME OF A CHANCE EVENT.

THE **elementary outcomes** ARE ALL POSSIBLE RESULTS OF THE RANDOM EXPERIMENT.

THE **sample space** IS THE SET OR COLLECTION OF ALL THE ELEMENTARY OUTCOMES.

IF THE EVENT WAS A COIN TOSS, FOR EXAMPLE, THE *RANDOM EXPERIMENT* CONSISTS OF RECORDING ITS OUTCOME...

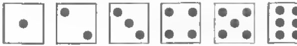
THE *ELEMENTARY OUTCOMES* ARE HEADS AND TAILS...



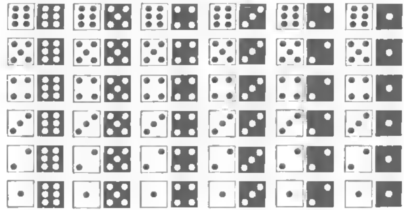
AND THE *SAMPLE SPACE* IS THE SET WRITTEN

$$\{H, T\}$$

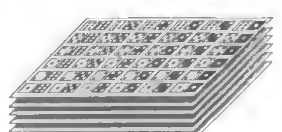
THE SAMPLE SPACE OF THE THROW OF A *SINGLE DIE* IS A LITTLE BIGGER



AND FOR A *PAIR OF DICE*, THE SAMPLE SPACE LOOKS LIKE THIS (WE MAKE ONE DIE WHITE AND ONE BLACK TO TELL THEM APART):

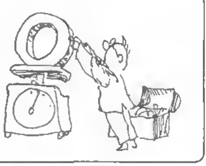


THIS SAMPLE SPACE HAS 36 (6X6) ELEMENTARY OUTCOMES. FOR THREE DICE, THE SPACE WOULD HAVE 216 ENTRIES, AS IN THIS 6X6X6 STACK. AND *FOUR DICE?*



AT SOME POINT, WE HAVE TO STOP LISTING, AND START THINKING...

NOW LET'S IMAGINE A RANDOM EXPERIMENT WITH n ELEMENTARY OUTCOMES $O_1, O_2, \dots O_n$. WE WANT TO ASSIGN A ***NUMERICAL WEIGHT***, OR ***PROBABILITY***, TO EACH OUTCOME, WHICH MEASURES THE LIKELIHOOD OF ITS OCCURRING. WE WRITE THE PROBABILITY OF O_i AS $P(O_i)$.



FOR EXAMPLE, IN A FAIR COIN TOSS, HEADS AND TAILS ARE EQUALLY LIKELY, AND WE ASSIGN THEM BOTH THE PROBABILITY .5.

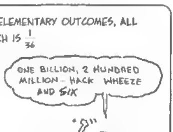
$P(H) = P(T) = .5$

EACH OUTCOME COMES UP HALF THE TIME. ASK ANY ***FOOTBALL PLAYER!***

IN THE ROLL OF *TWO DICE*, THERE ARE 36 ELEMENTARY OUTCOMES, ALL EQUALLY LIKELY, SO THE PROBABILITY OF EACH IS $\frac{1}{36}$

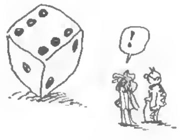
FOR INSTANCE,

$$P(\text{BLACK } 5, \text{ WHITE } 2) = \frac{1}{36}$$



WHICH MEANS IF YOU ROLLED THE DICE A VERY LARGE NUMBER OF TIMES, IN THE LONG RUN THIS OUTCOME WOULD OCCUR $\frac{1}{36}$ OF THE TIME.

WHAT IF OUR GAMBLER *CHEATS* AND THROWS A *LOADED DIE?* FOR THE SAKE OF ARGUMENT, SUPPOSE THAT NOW A ONE COMES UP *25%* OF THE TIME (IN THE LONG RUN).

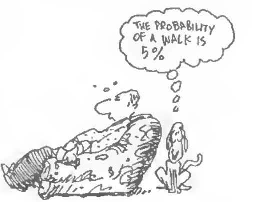


THE SAMPLE SPACE IS THE SAME AS FOR A FAIR DIE

$$\{1, 2, 3, 4, 5, 6\}$$

BUT THE PROBABILITIES ARE DIFFERENT. NOW $P(1) = .25$ AND THE REMAINING PROBABILTIES ADD UP TO 75. IF 2, 3, 4, 5, AND 6 WERE ALL EQUALLY LIKELY, THEN EACH ONE WOULD HAVE PROBABILITY $.15 = \frac{1}{5}(.75)$

IN GENERAL, ELEMENTARY OUTCOMES NEED NOT HAVE EQUAL PROBABILITY.



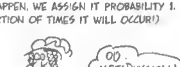
NOW WHAT CAN WE SAY ABOUT THE PROBABILITIES $P(O_i)$ IN AN ARBITRARY RANDOM EXPERIMENT? FIRST OF ALL,

$$P(O_i) \geq 0$$

PROBABILITIES ARE *NEVER NEGATIVE*. A PROBABILITY OF ZERO MEANS AN EVENT CAN'T HAPPEN. LESS THAN ZERO WOULD BE MEANINGLESS

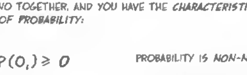
SECOND, IF AN EVENT IS *CERTAIN* TO HAPPEN, WE ASSIGN IT PROBABILITY 1. (IN THE LONG RUN, THAT'S THE PROPORTION OF TIMES IT WILL *OCCUR*!)

IN PARTICULAR, THE *TOTAL PROBABILITY OF THE SAMPLE SPACE* MUST BE 1. IF WE DO THE EXPERIMENT, *SOMETHING* IS BOUND TO HAPPEN!



PUT THESE TWO TOGETHER, AND YOU HAVE THE *CHARACTERISTIC PROPERTIES OF PROBABILITY:*

$P(O_i) \geq 0$	PROBABILITY IS *NON-NEGATIVE*
$P(O_1)+P(O_2)+\ldots+P(O_n)=1$	TOTAL PROBABILITY OF ALL ELEMENTARY OUTCOMES IS *ONE*.



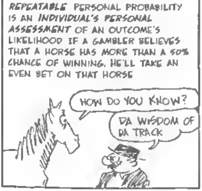
LIKE A CLEVER POLITICIAN, WE HAVE AVOIDED CERTAIN *UNPLEASANT QUESTIONS,* SUCH AS A) WHAT DOES PROBABILITY *MEAN?* AND B) *HOW DO WE ASSIGN* PROBABILITIES TO OUTCOMES?

HERE ARE SOME APPROACHES THAT HAVE BEEN TAKEN:

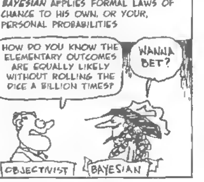
Classical PROBABILITY BASED ON GAMBLING IDEAS. THE FUNDAMENTAL ASSUMPTION IS THAT THE GAME IS FAIR AND ALL ELEMENTARY OUTCOMES HAVE THE SAME PROBABILITY.

Relative Frequency: WHEN AN EXPERIMENT *CAN* BE REPEATED, THEN AN EVENT'S PROBABILITY IS THE PROPORTION OF TIMES THE EVENT *OCCURS* IN THE LONG RUN

Personal PROBABILITY: MOST OF LIFE'S EVENTS ARE *NOT REPEATABLE.* PERSONAL PROBABILITY IS AN *INDIVIDUAL'S PERSONAL ASSESSMENT* OF AN OUTCOME'S LIKELIHOOD. IF A GAMBLER BELIEVES THAT A HORSE HAS MORE THAN A 50% CHANCE OF WINNING, HE'LL TAKE AN EVEN BET ON THAT HORSE



AN *OBJECTIVIST* USES EITHER THE CLASSICAL OR FREQUENCY DEFINITION OF PROBABILITY. A *SUBJECTIVIST* OR *BAYESIAN* APPLIES FORMAL LAWS OF CHANCE TO HIS OWN, OR YOUR, PERSONAL PROBABILITIES

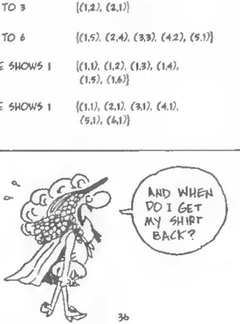


BASIC OPERATIONS

SO FAR, WE HAVE DISCUSSED ONLY THE PROBABILITY OF ELEMENTARY OUTCOMES. IN THEORY, THAT WOULD BE ENOUGH TO DESCRIBE ANY RANDOM EXPERIMENT, BUT IN PRACTICE IT'S PRETTY UNWIELDY. FOR EXAMPLE, EVEN SUCH AN ORDINARY OCCURRENCE AS ROLLING A SEVEN IS NOT AN ELEMENTARY OUTCOME. SO WE INTRODUCE A *NEW IDEA*:

AN ***EVENT*** IS A SET OF ELEMENTARY OUTCOMES. THE PROBABILITY OF AN EVENT IS THE SUM OF THE PROBABILITIES OF THE ELEMENTARY OUTCOMES IN THE SET. FOR INSTANCE, SOME EVENTS IN THE LIFE OF A TWO-DICED ROLLER ARE:

EVENT DESCRIPTION	EVENT'S ELEMENTARY OUTCOMES	PROBABILITY
A: DICE ADD TO 3	{(1,2), (2,1)}	$P(A) = \frac{2}{36}$
B: DICE ADD TO 6	{(1,5), (2,4), (3,3), (4,2), (5,1)}	$P(B) = \frac{5}{36}$
C: WHITE DIE SHOWS 1	{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)}	$P(C) = \frac{6}{36}$
D: BLACK DIE SHOWS 1	{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1)}	$P(D) = \frac{6}{36}$



THE BEAUTY OF USING EVENTS, RATHER THAN ELEMENTARY OUTCOMES, IS THAT WE CAN *COMBINE* EVENTS TO MAKE OTHER EVENTS, USING *LOGICAL OPERATIONS*. THE RELEVANT WORDS ARE ***AND, OR,*** AND ***NOT.***

THAT IS, GIVEN EVENTS E AND F, WE CAN MAKE NEW EVENTS.

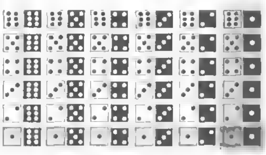
E **and** F : THE EVENT E AND THE EVENT F BOTH OCCUR.

E **or** F : THE EVENT E OR THE EVENT F OCCURS (OR BOTH DO).

not E THE EVENT E DOES NOT OCCUR

COMBINING OUR PRIMITIVE DEFINITIONS OF PROBABILITY WITH THESE LOGICAL OPERATIONS WILL GIVE US SOME POWERFUL FORMULAS FOR MANIPULATING PROBABILITIES

LET'S RETURN TO THE DICE-THROWING EXAMPLE. IF C IS THE EVENT, WHITE DIE = 1, AND D IS THE EVENT, BLACK DIE = 1, THEN.



C OR D IS THE ENTIRE SHADED AREA (WHERE ONE DIE OR THE OTHER IS 1).

C AND D IS WHERE THE SHADED AREAS OVERLAP (BOTH DICE ARE 1).

THIS ILLUSTRATES THE *ADDITION RULE:* FOR ANY EVENTS E, F,

$$P(E \text{ OR } F) = P(E) + P(F) - P(E \text{ AND } F)$$

ADDING $P(E) + P(F)$ *DOUBLE COUNTS* THE ELEMENTARY OUTCOMES SHARED BY E AND F, SO WE HAVE TO SUBTRACT THE EXTRA AMOUNT, WHICH IS $P(E \text{ AND } F)$.

IN THE ABOVE EXAMPLE.

$$P(C \text{ OR } D) = \frac{11}{36}$$

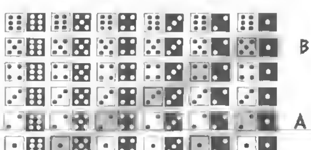
AS YOU CAN SEE BY COUNTING ELEMENTARY OUTCOMES. LIKEWISE,

$$P(C \text{ AND } D) = \frac{1}{36}$$

AND WE CONFIRM THE FORMULA:

$$P(C) + P(D) - P(C \text{ AND } D)$$
$$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}$$
$$= P(C \text{ OR } D)$$

SOMETIMES, THE OVERLAP E AND F IS EMPTY, AND THE TWO EVENTS HAVE NO ELEMENTARY OUTCOMES IN COMMON. IN THAT CASE, WE SAY E AND F ARE ***MUTUALLY EXCLUSIVE,*** MAKING P(E AND F) = 0. HERE WE SEE THE MUTUALLY EXCLUSIVE EVENTS **A**, THE DICE ADD TO 3, AND **B**, THE DICE ADD TO 6.



FOR MUTUALLY EXCLUSIVE EVENTS, WE GET A ***SPECIAL ADDITION RULE:*** IF E AND F ARE MUTUALLY EXCLUSIVE, THEN

P(E OR F) = P(E) + P(F)

AND WE CHECK THAT P(A OR B) = $\frac{7}{36} = \frac{2}{36} + \frac{5}{36}$ = P(A) + P(B)

AND FINALLY, A ***SUBTRACTION RULE:*** FOR ANY EVENT E,

P(E) = 1 − P(NOT E)

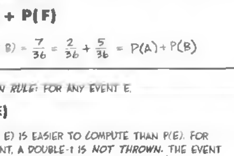
THIS IS USEFUL WHEN P(NOT E) IS EASIER TO COMPUTE THAN P(E). FOR INSTANCE, LET E BE THE EVENT, A DOUBLE-1 IS *NOT THROWN*. THE EVENT NOT-E, A DOUBLE-1 *IS* THROWN, HAS PROBABILITY P(NOT E) = $\frac{1}{36}$

SO

P(E) = 1 − P(NOT E)

$= 1 - \frac{1}{36}$

$= \frac{35}{36}$

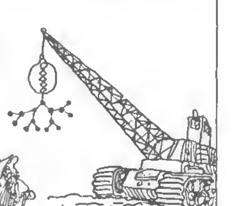


THE FORMULAS WE JUST DERIVED ARE, IN FACT, ADEQUATE FOR ANSWERING DE MERE'S QUESTION— BUT NOT EASILY! (YOU MIGHT TRY USING THEM ON A SIMPLER QUESTION. WHAT'S THE PROBABILITY OF ROLLING AT LEAST ONE SIX IN TWO ROLLS OF A SINGLE DIE?) WE NEED *MORE MACHINERY!*

SO WE INTRODUCE

conditional probability

(AN ESSENTIAL CONCEPT IN STATISTICS!)



SUPPOSE WE ALTER OUR EXPERIMENT SLIGHTLY, AND THROW THE WHITE DIE *BEFORE* THE BLACK DIE. WHAT'S THE PROBABILITY THAT THE FACES SUM TO 3?

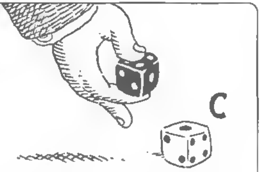
BEFORE THE DICE ARE THROWN, THE PROBABILITY IS $P(A)=\frac{2}{36}$

NOW SUPPOSE THE WHITE DIE COMES UP 1 (EVENT C). WHAT'S THE PROBABILITY OF A NOW?

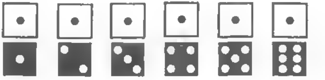
WE CALL IT THE *CONDITIONAL PROBABILITY* THAT EVENT A WILL OCCUR, GIVEN THE *CONDITION* THAT EVENT C HAS ALREADY OCCURRED. WE WRITE

P(A|C)

AND SAY "THE PROBABILITY OF A, *GIVEN C*."



BEFORE ANY DICE WERE THROWN, THE SAMPLE SPACE HAD 36 OUTCOMES, BUT NOW THAT THE EVENT C HAS OCCURRED, THE OUTCOME MUST BELONG TO THE *REDUCED SAMPLE SPACE C*.



IN THE REDUCED SAMPLE SPACE OF SIX ELEMENTARY OUTCOMES, ONLY ONE OUTCOME (1,2) SUMS TO 3. SO THE CONDITIONAL PROBABILITY IS 1/6.

IN GENERAL, TO FIND THE CONDITIONAL PROBABILITY P(E|F), WE LOOK AT THE EVENT E AND F AS PART OF THE REDUCED SAMPLE SPACE F.

WE TRANSLATE THIS INTO A FORMAL DEFINITION: THE *CONDITIONAL PROBABILITY OF E, GIVEN F,* IS

$$P(E|F) = \frac{P(E \text{ and } F)}{P(F)}$$

FROM WHICH YOU CAN DIRECTLY VERIFY SOME INTUITIVE FACTS.

$P(E|E) = 1$ (ONCE E OCCURS, IT'S CERTAIN)

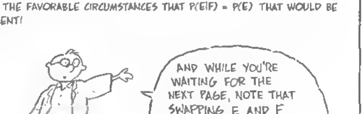
WHEN E AND F ARE MUTUALLY EXCLUSIVE,

$P(E|F) = 0$ (ONCE F HAS OCCURRED, E IS IMPOSSIBLE)

REARRANGING THE DEFINITION GIVES US THE *MULTIPLICATION RULE:*

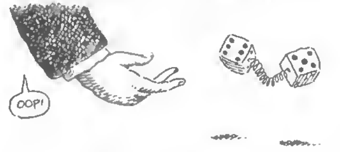
$$P(E \text{ AND } F) = P(E|F)P(F)$$

WHICH WE WOULD LIKE TO REDUCE TO A "SPECIAL" MULTIPLICATION RULE, UNDER THE FAVORABLE CIRCUMSTANCES THAT $P(E|F) = P(E)$ THAT WOULD BE EXCELLENT!



INDEPENDENCE and the special multiplication rule.

TWO EVENTS E AND F ARE *INDEPENDENT* OF EACH OTHER IF THE OCCURRENCE OF ONE HAS *NO INFLUENCE* ON THE PROBABILITY OF THE OTHER. FOR INSTANCE, THE ROLL OF ONE DIE HAS NO EFFECT ON THE ROLL OF ANOTHER (UNLESS THEY'RE GLUED TOGETHER, MAGNETIC, ETC.!).



IN TERMS OF CONDITIONAL PROBABILITY, THIS AMOUNTS TO SAYING $P(E) = P(E|F)$ OR, EQUIVALENTLY, $P(F) = P(F|E)$. WHEN E AND F ARE INDEPENDENT, WE GET A *SPECIAL MULTIPLICATION RULE:*

$$\mathbf{P(E \text{ AND } F) = P(E)P(F)}$$

LET'S VERIFY THE INDEPENDENCE OF DICE, USING THE FORMULAS. C IS THE EVENT *WHITE DIE COMES UP 1;* D IS THE EVENT *BLACK DIE COMES UP 1,* AND WE HAVE:

$$P(C|D) = \frac{P(C \text{ AND } D)}{P(D)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6} = P(C)$$

BUT THE WHITE DIE SHOWING 1 OBVIOUSLY DOES AFFECT THE CHANCES THAT THE SUM OF THE TWO DICE IS 3!

$$P(A|C) = \frac{P(A \text{ AND } C)}{P(C)} = \frac{P(1,2)}{P(C)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6} \neq P(A) = \frac{1}{18}$$

SO THESE TWO EVENTS ARE *NOT INDEPENDENT.*

BEFORE GOING ON, LET'S SUMMARIZE ALL THE RULES WE'VE ACCUMULATED:

ADDITION RULE:

P(E OR F) = P(E) + P(F) - P(E AND F)

SPECIAL ADDITION RULE. WHEN E AND F ARE MUTUALLY EXCLUSIVE,

P(E OR F) = P(E) + P(F)

SUBTRACTION RULE.

P(E) = 1 - P(NOT E)

MULTIPLICATION RULE.

P(E AND F) = P(E|F)P(F)

SPECIAL MULTIPLICATION RULE WHEN E AND F ARE INDEPENDENT,

P(E AND F) = P(E)P(F)

AND NOW, DE MERE'S PROBLEM AT LAST... LET E BE THE EVENT OF GETTING AT LEAST ONE SIX IN FOUR ROLLS OF A SINGLE DIE. WHAT'S P(E)? THIS IS ONE OF THOSE EVENTS WHOSE NEGATIVE IS EASIER TO DESCRIBE: *NOT E* IS THE EVENT OF *GETTING NO SIXES IN FOUR THROWS.*

IF A_i IS THE EVENT, GETTING NO SIX ON THE i^{TH} THROW, WE KNOW THAT $P(A_i) = \frac{5}{6}$. WE ALSO KNOW THAT ROLLS ARE INDEPENDENT, SO

$P(\text{NOT } E) =$

$P(A_1 \text{ AND } A_2 \text{ AND } A_3 \text{ AND } A_4)$

MULTIPLICATION RULE $\longrightarrow$ $= \left(\frac{5}{6}\right)^4 = .482,$

SO

$P(E) = 1 - P(\text{NOT } E) = .518$

NOW THE SECOND HALF: LET F BE THE EVENT, GETTING AT LEAST ONE DOUBLE SIX IN 24 THROWS. AGAIN, NOT F IS EASIER TO DESCRIBE. IT'S THE EVENT OF GETTING NO DOUBLE SIXES.

IF B_i IS THE EVENT, NO DOUBLE SIX IS THROWN ON THE i^{TH} ROLL, THEN NOT F = B_1 AND B_2 AND... B_{24}. THE PROBABILITY OF EACH B IS

$$P(B_i) = \frac{35}{36}, \text{ SO}$$

$$P(\text{NOT } F) = \left(\frac{35}{36}\right)^{24} = .509$$

(BY THE MULTIPLICATION RULE) AND WE CONCLUDE THAT

$$P(F) = 1 - P(\text{NOT } F) = 1 - .509$$

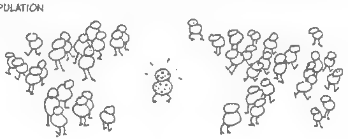
$$= .491$$

DE MERE TOLD PASCAL HE HAD ACTUALLY OBSERVED THAT EVENT F OCCURRED LESS OFTEN THAN EVENT E, BUT HE WAS AT A LOSS TO EXPLAIN WHY... FROM WHICH WE CONCLUDE THAT DE MERE GAMBLED OFTEN AND *KEPT CAREFUL RECORDS!!*

BAYES THEOREM and the case of the false positives...

FOR A MORE SERIOUS APPLICATION OF CONDITIONAL PROBABILITY, LET'S ENTER AN ARENA OF LIFE AND DEATH...

SUPPOSE A RARE DISEASE INFECTS ONE OUT OF EVERY 1000 PEOPLE IN A POPULATION



AND SUPPOSE THAT THERE IS A GOOD, BUT NOT PERFECT, TEST FOR THIS DISEASE: IF A PERSON HAS THE DISEASE, THE TEST COMES BACK POSITIVE 99% OF THE TIME. ON THE OTHER HAND, THE TEST ALSO PRODUCES SOME *FALSE POSITIVES*. ABOUT 2% OF UNINFECTED PATIENTS ALSO TEST POSITIVE. AND YOU JUST TESTED POSITIVE. WHAT ARE YOUR CHANCES OF HAVING THE DISEASE?

WE HAVE TWO EVENTS TO WORK WITH.

A : PATIENT HAS THE DISEASE
B . PATIENT TESTS POSITIVE.

THE INFORMATION ABOUT THE TEST'S EFFECTIVENESS CAN BE WRITTEN

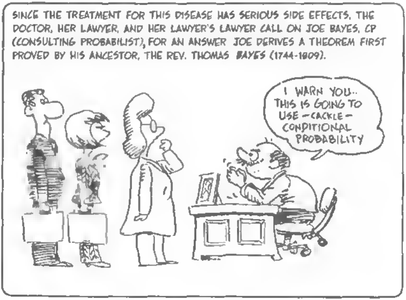
$P(A) = .001$ (ONE PATIENT IN 1000 HAS THE DISEASE)

$P(B|A) = .99$ (PROBABILITY OF A POSITIVE TEST, GIVEN INFECTION, IS .99)

$P(B|\text{NOT } A) = .02$ (PROBABILITY OF A FALSE POSITIVE, GIVEN NO INFECTION, IS .02)

AND WE ASK

$P(A|B) = \text{WHAT?}$ (PROBABILITY OF HAVING THE DISEASE, GIVEN A POSITIVE TEST)



JOE BEGINS WITH A 2X2 TABLE, WHICH DIVIDES THE SAMPLE SPACE INTO FOUR MUTUALLY EXCLUSIVE EVENTS. IT DISPLAYS EVERY POSSIBLE COMBINATION OF DISEASE STATE AND TEST RESULT.

	A	NOT A
B	A AND B	NOT A AND B
NOT B	A AND NOT B	NOT A AND NOT B

LET'S FIND THE PROBABILITIES OF EACH EVENT IN THE TABLE:

	A	NOT A	SUM
B	P(A AND B)	P(NOT A AND B)	P(B)
NOT B	P(A AND NOT B)	P(NOT A AND NOT B)	P(NOT B)
	P(A)	P(NOT A)	1

THE PROBABILITIES IN THE MARGINS ARE FOUND BY SUMMING ACROSS ROWS AND DOWN COLUMNS.

NOW COMPUTE:

$$P(A \text{ AND } B) = P(B|A)P(A) = (.99)(.001) = .00099$$

$$P(\text{NOT } A \text{ AND } B) = P(B|\text{NOT } A)P(\text{NOT } A) = (.02)(.999) = .01998$$

ALLOWING US TO FILL IN SOME ENTRIES

	A	NOT A	SUM
B	00099	.01998	.02097
NOT B	P(A AND NOT B)	P(NOT A AND NOT B)	P(NOT B)
	.001	999	1

WE FIND THE REMAINING PROBABILITIES BY SUBTRACTING IN THE COLUMNS, THEN ADDING ACROSS THE ROWS.

THE FINAL TABLE IS

	A	NOT A		
B	.00099	.01998	.02097	P(B)
NOT B	.00001	.97902	.97903	P(NOT B)
	.001	.999	1	
	P(A)	P(NOT A)		

FROM WHICH WE DIRECTLY DERIVE

$$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)} = \frac{.00099}{.02097} = \mathbf{.0472}$$

DESPITE THE HIGH ACCURACY OF THE TEST, *LESS THAN 5%* OF THOSE WHO TEST POSITIVE ACTUALLY HAVE THE DISEASE! THIS IS CALLED THE *FALSE POSITIVE PARADOX.*

THIS TABLE SHOWS WHAT HAPPENS IN A GROUP OF A THOUSAND PATIENTS. ON AVERAGE, ONLY 21 PEOPLE WILL TEST POSITIVE--AND ONLY *ONE* OF THEM HAS THE DISEASE! 20 FALSE POSITIVES COME FROM THE *MUCH LARGER UNINFECTED GROUP.*

	DISEASE	NO DISEASE	
TESTS POSITIVE	1	20	21
TESTS NEGATIVE	0	979	979
	1	999	1000

WHAT'S THE PHYSICIAN TO DO? JOE BAYES ADVISES HER NOT TO START TREATMENT ON THE BASIS OF THIS TEST ALONE. THE TEST DOES PROVIDE INFORMATION, HOWEVER: WITH A POSITIVE TEST THE PATIENT'S CHANCE OF HAVING THE DISEASE INCREASED FROM 1 IN 1000 TO 1 IN 23. THE DOCTOR FOLLOWS UP WITH MORE TESTS.

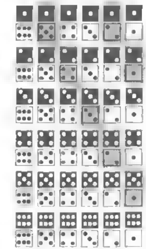
JOE BAYES COLLECTS HIS CONSULTING CHECK BEFORE ADMITTING THAT ALL THOSE STEPS HE WENT THROUGH CAN BE COMPRESSED INTO THE SINGLE FORMULA CALLED BAYES THEOREM

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A)+P(\text{NOT } A)P(B|\text{NOT } A)}$$

IT COMPUTES $P(A|B)$ FROM $P(A)$ AND THE TWO CONDITIONAL PROBABILITIES $P(B|A)$ AND $P(B|\text{NOT } A)$. YOU CAN DERIVE IT BY NOTING THAT THE BIG FRACTION CAN BE EXPRESSED AS

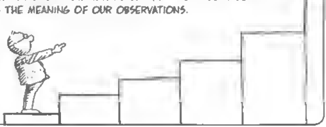
$$\frac{P(A \text{ and } B)}{P(A \text{ and } B)+P(\text{NOT } A \text{ and } B)} = \frac{P(A \text{ and } B)}{P(B)} = P(A|B)$$

IN THIS CHAPTER, WE COVERED THE BASICS OF PROBABILITY: ITS DEFINITION, SAMPLE SPACES AND ELEMENTARY OUTCOMES, CONDITIONAL PROBABILITY, AND SOME BASIC FORMULAS FOR COMPUTING PROBABILITIES. WE ILLUSTRATED THESE IDEAS USING A 2-DICE SAMPLE SPACE. FOR THE MODERN GAMBLER, PROBABILITY IS THE POWER TOOL OF CHOICE.



AND FINALLY, IN THE MEDICAL EXAMPLE, WE SHOWED HOW THESE ABSTRACT IDEAS COULD HELP TO MAKE GOOD DECISIONS IN THE FACE OF *IMPERFECT INFORMATION* AND *REAL RISKS*—THE *ULTIMATE GOAL* OF STATISTICS.

BUT THIS IS JUST THE BEGINNING FOR US. PROBABILITY IS ONLY A *TOOL*—AN ESSENTIAL TOOL, TO BE SURE—IN THE STUDY OF STATISTICS. IN THE CHAPTERS THAT FOLLOW, WE'LL EXPLORE THE SUBTLE RELATIONSHIP BETWEEN PROBABILITY, VARIATIONS IN STATISTICAL DATA, AND OUR CONFIDENCE IN INTERPRETING THE MEANING OF OUR OBSERVATIONS.



•Chapter 4•

RANDOM VARIABLES

IN CHAPTER 2, WE SAW THAT OBSERVATIONS OF NUMERICAL DATA, LIKE STUDENTS' WEIGHTS, CAN BE GRAPHED AND SUMMARIZED IN TERMS OF MIDPOINTS, SPREADS, OUTLIERS, ETC. IN CHAPTER 3, WE SAW HOW PROBABILITIES CAN BE ASSIGNED TO THE OUTCOMES OF A RANDOM EXPERIMENT.

IF WE IMAGINE A RANDOM EXPERIMENT REPEATED MANY TIMES, WE EXPECT THAT THE ACTUAL OUTCOMES OVER TIME WILL BE GOVERNED BY THEIR PROBABILITIES. THE PROBABILITIES FORM A ***MODEL*** FOR REAL-LIFE EXPERIMENTS... SO WHY NOT DO FOR THE MODEL WHAT WE'VE ALREADY DONE FOR THE DATA IT DESCRIBES?

THE KEY IDEA IS THE *RANDOM VARIABLE*, WHICH WE WRITE AS A LARGE

X

A RANDOM VARIABLE IS DEFINED AS THE *NUMERICAL OUTCOME OF A RANDOM EXPERIMENT.*

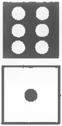
FOR EXAMPLE, IMAGINE DRAWING ONE STUDENT AT RANDOM FROM THE STUDENT BODY. THAT'S THE RANDOM EXPERIMENT. THE STUDENT'S *HEIGHT, WEIGHT, FAMILY INCOME, S.A.T. SCORE, AND GRADE POINT AVERAGE* ARE ALL NUMERICAL *VARIABLES* DESCRIBING PROPERTIES OF THE RANDOMLY SELECTED STUDENT. THEY'RE ALL *RANDOM VARIABLES*.

ANOTHER EXAMPLE: TOSS TWO COINS (THE RANDOM EXPERIMENT) AND RECORD THE *NUMBER* OF HEADS: 0, 1, OR 2

OUTCOME	TT	HT TH	HH
x	0	1	2

NOTE THE NOTATION! THE VARIABLE IS WRITTEN WITH A CAPITAL X. THE LOWERCASE x REPRESENTS A SINGLE VALUE OF X, FOR EXAMPLE $x=2$, IF HEADS COMES UP TWICE.

ANOTHER EXAMPLE IS BASED ON THE FAMILIAR TOSS OF TWO DICE. LET Y REPRESENT THE SUM OF THE DOTS ON THE TWO DICE. FOR THIS RANDOM VARIABLE, Y CAN BE ANY NUMBER BETWEEN 2 AND 12.



$Y=7$

NOW WE WANT TO LOOK AT THE ***PROBABILITIES*** OF THE OUTCOMES. FOR THE PROBABILITY THAT THE RANDOM VARIABLE X HAS THE VALUE x, WE WRITE $Pr(X = x)$, OR JUST $p(x)$. FOR THE COIN-FLIPPING RANDOM VARIABLE X, WE CAN MAKE THE TABLE:

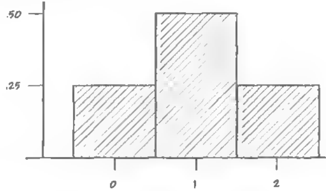
x	0	1	2
$Pr(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

THIS TABLE IS CALLED THE ***PROBABILITY DISTRIBUTION*** OF THE RANDOM VARIABLE X.

FOR THE RANDOM VARIABLE Y (THE SUM OF TWO DICE), THE PROBABILITY DISTRIBUTION LOOKS LIKE THIS.

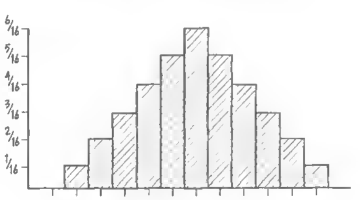
y	2	3	4	5	6	7	8	9	10	11	12
$Pr(Y=y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

NOW LET'S DRAW GRAPHS, OR ***HISTOGRAMS,*** SHOWING THESE PROBABILITY DISTRIBUTIONS. FOR EACH VALUE OF X, WE DRAW A BAR EQUAL IN HEIGHT TO $p(x)$.

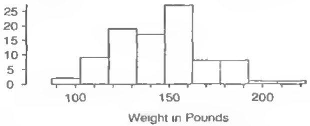


IT'S EASY TO SEE THAT THE TOTAL AREA OF THESE BOXES IS 1: EACH BOX HAS BASE 1 AND HEIGHT $p(x)$, SO THE TOTAL AREA IS THE SUM OF THE PROBABILITIES OF ALL OUTCOMES, I.E. 1.

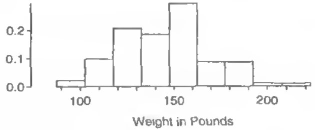
HERE'S THE PROBABILITY HISTOGRAM OF THE RANDOM VARIABLE Y, SHOWING THE PROBABILITY DISTRIBUTION OF THE SUM OF TWO DICE:



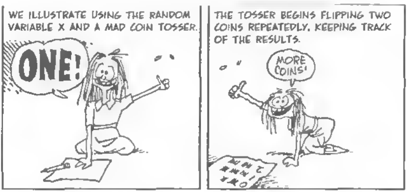
WHY DO WE CALL THESE GRAPHS HISTOGRAMS? YOU'LL RECALL THAT IN CHAPTER 2, A HISTOGRAM WAS A GRAPH THAT DISPLAYED HOW MANY DATA POINTS LAY IN EACH OF A SERIES OF INTERVALS



FROM THIS *FREQUENCY* HISTOGRAM, WE DERIVED THE *RELATIVE* FREQUENCY HISTOGRAM, SHOWING THE *PROPORTION* OF DATA IN EACH INTERVAL:



BUT YOU'LL RECALL THAT, BY ONE DEFINITION, PROBABILITY IS THE *RELATIVE FREQUENCY OF AN EVENT "IN THE LONG RUN."* IF WE REPEAT THE RANDOM EXPERIMENT MANY TIMES, THE *RELATIVE FREQUENCY* HISTOGRAM OF THE OUTCOMES SHOULD COME TO LOOK VERY MUCH LIKE THE RANDOM VARIABLE'S *PROBABILITY* HISTOGRAM!

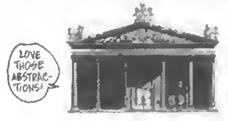


WE KNOW X'S PROBABILITY DISTRIBUTION, AND WE ALSO KNOW THAT THE ACTUAL COIN FLIPS WILL MATCH THE PROBABILITIES APPROXIMATELY AFTER 1000 TOSSES. THE MAD TOSSER TALLIES HER DATA.

PROBABILITY MODEL		OBSERVED DATA	
$p(x)$	x	n_x = NUMBER OF OCCURRENCES	$\frac{n_x}{n}$ = RELATIVE FREQUENCY
.25	0	260	.260
.5	1	517	.517
.25	2	223	.223

AND WE SEE THAT THE PROBABILITY HISTOGRAM OF X LOOKS LIKE THE "PURE FORM" OR MODEL OF THE RELATIVE FREQUENCY HISTOGRAM OF THE DATA.

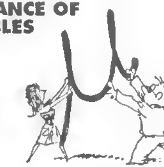
TO EXTEND THE ANALOGY BETWEEN RELATIVE FREQUENCY AND DATA, WE SHOULD NOW BE WILLING TO TALK ABOUT THE MEAN AND VARIANCE (OR STANDARD DEVIATION) OF A PROBABILITY DISTRIBUTION..



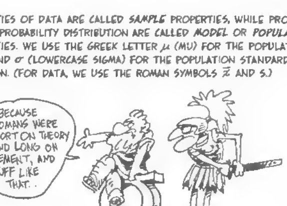
AND JUST TO REMIND OURSELVES THAT WE'RE IN THE REALM OF THE ABSTRACT, WE BREAK OUT SOME *GREEK LETTERS...*

MEAN AND VARIANCE OF RANDOM VARIABLES

WE USE SPECIAL TERMINOLOGY AND SYMBOLS TO DISTINGUISH BETWEEN THE PROPERTIES OF DATA SETS AND PROBABILITY DISTRIBUTIONS.



PROPERTIES OF DATA ARE CALLED *SAMPLE* PROPERTIES, WHILE PROPERTIES OF THE PROBABILITY DISTRIBUTION ARE CALLED *MODEL* OR *POPULATION* PROPERTIES. WE USE THE GREEK LETTER μ (MU) FOR THE POPULATION MEAN, AND σ (LOWERCASE SIGMA) FOR THE POPULATION STANDARD DEVIATION. (FOR DATA, WE USE THE ROMAN SYMBOLS $\bar{x}$ AND s.)



THE SAMPLE MEAN WAS DEFINED BY THE EQUATION

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

NOW SOME OF THESE DATA POINTS x_i MAY WELL HAVE EQUAL VALUES. THINK OF THE MAD COIN TOSSER: THE ONLY AVAILABLE VALUES WERE 0, 1, AND 2, AND SHE MADE 1000 TOSSES. THE VALUE 0 WAS TAKEN ON 260 TIMES, 1 HEAD CAME UP 517 TIMES, AND 2 HEADS, 223 TIMES.

AS WE LET x RANGE OVER ALL VALUES OF X, CALL n_x THE NUMBER OF DATA POINTS WITH THE VALUE x. THEN WE CAN REWRITE THAT FORMULA AS

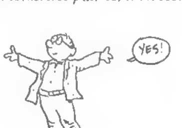
$$\bar{x} = \frac{1}{n}\sum_{\text{all } x} n_x x$$

OR

$$\bar{x} = \sum_{\text{all } x} x \frac{n_x}{n}$$

AH! BUT NOW $\frac{n_x}{n}$ IS THE RELATIVE FREQUENCY... THE "APPROXIMATE PROBABILITY..." THE NUMBER THAT APPROACHES $p(x)$...SO, BY ANALOGY, WE FORM THE EXPRESSION

$$\sum_{\text{all } x} x\,p(x)$$



AND DEFINE THAT AS THE ***MEAN OF THE PROBABILITY DISTRIBUTION.***

DEFINITION: THE **mean** OF THE RANDOM VARIABLE X IS DEFINED AS

$$\mu = \sum_{\text{all } x} x p(x)$$

THIS IS ALSO CALLED THE ***EXPECTED VALUE*** OF X, OR E[X]. THINK OF IT AS THE SUM OF THE POSSIBLE VALUES, EACH WEIGHTED BY ITS PROBABILITY.

THE MAD COIN TOSSER'S EXPERIMENT ALLOWS US TO COMPARE HER SAMPLE MEAN $\bar{x}$ WITH OUR MODEL MEAN μ:

SAMPLE

x	$\frac{n_x}{n}$	$x\frac{n_x}{n}$
0	.26	0
1	.517	.517
2	.223	.446
		$.963 = \bar{x}$

MODEL

x	$p(x)$	$xp(x)$
0	.25	0
1	.5	.5
2	.25	.5
		$1 = \mu$

NOW LET'S DO THE SAME THING TO THE ***VARIANCE***. MAYBE YOU REMEMBER THE FORMULA

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

IT (ALMOST) MEASURES THE AVERAGE SQUARED DISTANCE OF DATA FROM THE MEAN. AS ABOVE THIS CAN BE REWRITTEN:

$$s^2 = \sum_{\text{all } x}(x-\bar{x})^2\frac{n_x}{n-1}$$

EXCEPT FOR THAT ANNOYING DENOMINATOR $n-1$ INSTEAD OF n, THIS ALSO LOOKS LIKE A WEIGHTED SUM OF SQUARED DISTANCES... SO WE MAKE ANOTHER *DEFINITION:*

THE **variance** OF A RANDOM VARIABLE X IS THE EXPECTED SQUARED DISTANCE FROM THE POPULATION MEAN:

$$\sigma^2 = \sum_{\text{all } x} (x-\mu)^2 p(x)$$

THE **standard deviation** σ IS THE SQUARE ROOT OF THE VARIANCE

WE USE THE TABLE FROM THE LAST PAGE TO FIND THE VARIANCE OF THE TWO-COIN TOSS (FOR WHICH $\mu = 1$).

x	$p(x)$	$(x-\mu)^2 p(x)$
0	.25	$(0-1)^2 .25 = .25$
1	.5	$(1-1)^2 .50 = 0$
2	.25	$(2-1)^2 .25 = .25$
TOTAL		$.50 = \sigma^2$

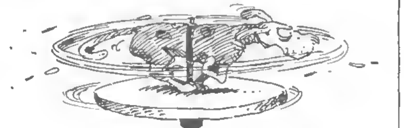
TO SUM UP μ AND σ, THE POPULATION *MEAN* AND *STANDARD DEVIATION*, ARE PROPERTIES WE CAN COMPUTE FROM *PROBABILITY DISTRIBUTIONS*. THEY ARE COMPLETELY ANALOGOUS TO THE SAMPLE MEAN $\bar{x}$ AND STANDARD DEVIATION s COMPUTED FROM SAMPLE DATA

OUR EXAMPLES SO FAR HAVE BEEN *DISCRETE* RANDOM VARIABLES. THEIR OUTCOMES ARE A SET OF ISOLATED ("DISCRETE") VALUES, LIKE THOSE WE SAW IN CHAPTER 3, BUT THERE ARE ALSO

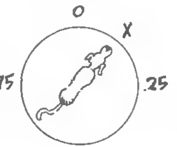
Continuous Random Variables

LET'S IMAGINE A RANDOM EXPERIMENT IN WHICH ***ALL OUTCOMES HAVE PROBABILITY ZERO.*** THAT'S RIGHT, $p(x) = 0$ FOR EVERY x.

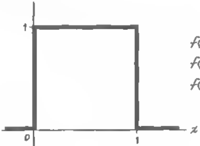
A SIMPLE EXAMPLE IS A BALANCED, SPINNING POINTER. IT CAN STOP ANYWHERE IN THE CIRCLE. IF X REPRESENTS THE PROPORTION OF THE TOTAL CIRCUMFERENCE IT LANDS ON, THE RANDOM VARIABLE X CAN TAKE ON ANY VALUE BETWEEN 0 AND 1—AN *INFINITE* RANGE OF VALUES.



SOME PROBABILITIES ARE EASY TO FIND, LIKE THE PROBABILITY THAT X FALLS WITHIN A RANGE. FOR EXAMPLE, $Pr(.25 \leq X \leq .75) = .5$, BECAUSE IT'S HALF THE CIRCLE. BUT WHAT ABOUT $Pr(X = .5)$? SINCE X CAN TAKE ON AN INFINITE NUMBER OF VALUES, AND ALL OF THESE VALUES ARE EQUALLY LIKELY, THE PROBABILITY THAT X IS EXACTLY .5 (OR EXACTLY ANYTHING) IS PRECISELY 0.



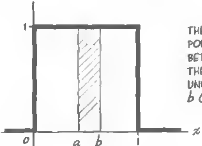
HOW CAN WE DRAW A PICTURE OF THIS? BY ANALOGY WITH THE CASE OF DISCRETE PROBABILITIES, WE TRY TO SEE CONTINUOUS PROBABILITIES AS ***AREAS UNDER SOMETHING.*** FOR THE SPINNING POINTER, THE "SOMETHING" LOOKS LIKE THIS.



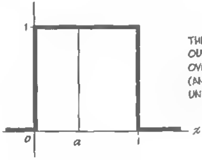
$f(x) = 0$ WHEN $x < 0$

$f(x) = 1$ WHEN $0 \leq x \leq 1$

$f(x) = 0$ WHEN $x > 1$

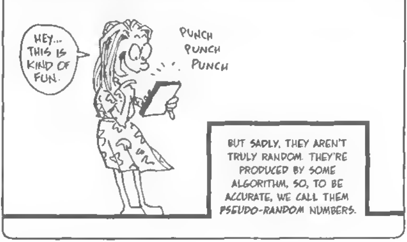


THE PROBABILITY THAT THE POINTER POINTS ANYWHERE BETWEEN a AND b IS PRECISELY THE AREA OF THE SHADED REGION UNDER THE CURVE BETWEEN a AND b (IN THIS CASE, $b-a$).

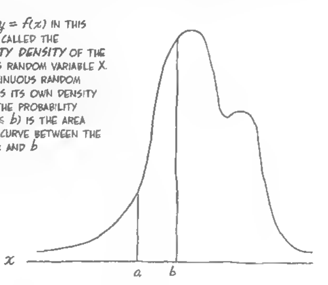


THE PROBABILITY OF AN EXACT OUTCOME, HOWEVER, IS THE "AREA" OVER A POINT, WHICH IS *ZERO.* (AND NOTE THAT THE TOTAL AREA UNDER THE CURVE IS EXACTLY 1.)

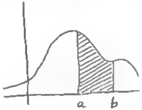
THE SAME PICTURE DESCRIBES THE *RANDOM NUMBER GENERATOR* FOUND ON MOST COMPUTERS AND SOME CALCULATORS. PRESS THE BUTTON: OUT POPS A NUMBER BETWEEN 0 AND 1, AND ALL THE NUMBERS ARE EQUALLY LIKELY, JUST AS WITH THE SPINNING POINTER.



THE CURVE $y = f(x)$ IN THIS EXAMPLE IS CALLED THE ***PROBABILITY DENSITY*** OF THE CONTINUOUS RANDOM VARIABLE X. EVERY CONTINUOUS RANDOM VARIABLE HAS ITS OWN DENSITY FUNCTION. THE PROBABILITY $Pr(a \leq X \leq b)$ IS THE AREA UNDER THE CURVE BETWEEN THE x-VALUES a AND b



IN GENERAL, THE PROBABILITY DENSITY WON'T BE SO SIMPLE, AND COMPUTING THE AREAS CAN BE FAR FROM TRIVIAL.



$$\int_a^b f(x)\,dx$$

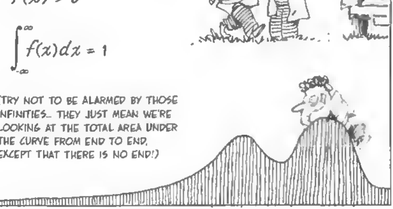
WE HAVE TO USE CALCULUS NOTATION TO DESCRIBE THE AREA UNDER THE CURVE $f(x)$. THIS SYMBOL IS READ "THE INTEGRAL OF f FROM a TO b"

LIKE DISCRETE PROBABILITIES, CONTINUOUS DENSITIES HAVE TWO FAMILIAR PROPERTIES.

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

(TRY NOT TO BE ALARMED BY THOSE INFINITIES... THEY JUST MEAN WE'RE LOOKING AT THE TOTAL AREA UNDER THE CURVE FROM END TO END, EXCEPT THAT THERE IS NO END!)



ALTHOUGH THE NOTATION MAY BE UNFAMILIAR, ALL IT MEANS IS AN *AREA*. THE INTEGRAL SIGN ITSELF IS A STRETCHED "S," FOR SUM, WHICH THE INTEGRAL, IN SOME SENSE, IS.

AS A SUMLIKE SOMETHING, THE INTEGRAL SERVES TO DEFINE THE

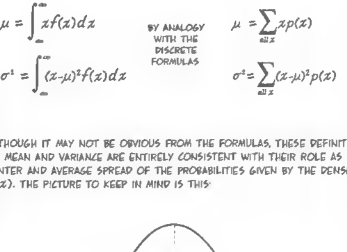
MEAN AND VARIANCE of a continuous random variable.

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx \qquad \mu = \sum_{\text{all } x} x p(x)$$

BY ANALOGY WITH THE DISCRETE FORMULAS

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \qquad \sigma^2 = \sum_{\text{all } x} (x-\mu)^2 p(x)$$

ALTHOUGH IT MAY NOT BE OBVIOUS FROM THE FORMULAS, THESE DEFINITIONS OF MEAN AND VARIANCE ARE ENTIRELY CONSISTENT WITH THEIR ROLE AS CENTER AND AVERAGE SPREAD OF THE PROBABILITIES GIVEN BY THE DENSITY $f(x)$. THE PICTURE TO KEEP IN MIND IS THIS:



ADDING
random variables

ONCE YOU KNOW THE MEAN AND VARIANCE OF A RANDOM VARIABLE, WHAT CAN YOU DO WITH THEM? WELL, FOR ONE THING, YOU CAN FIND THE MEAN AND VARIANCE OF SOME *OTHER* RANDOM VARIABLES...

FOR EXAMPLE, LOOK AT A FAIR COIN TOSS LET X = 1 IF THE COIN COMES UP HEADS AND 0 IF IT COMES UP TAILS.

x	0	1
$p(x)$	.5	.5

BY NOW, YOU SHOULD BE ABLE TO FIND THE MEAN

$$E[X] = 0\cdot p(0) = 1\cdot p(1)$$
$$= 0 + .5$$
$$= .5$$

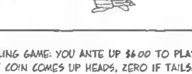
AND THE VARIANCE

$$\sigma^2 = (0-.5)^2 p(0) + (1-.5)^2 p(1)$$
$$= .25$$

NOW LET'S PLAY A SIMPLE GAMBLING GAME: YOU ANTE UP $6.00 TO PLAY; I FLIP A COIN; YOU WIN $10 IF THE COIN COMES UP HEADS, ZERO IF TAILS. THEN YOUR WINNINGS **W** ARE

$$W = 10X - 6$$

A NEW RANDOM VARIABLE! WHAT ARE ITS MEAN AND VARIANCE?



A LITTLE THOUGHT SHOULD CONVINCE YOU THAT E[W] IS GIVEN BY

$$E[W] = E[10X - 6] = 10E[X] - 6$$

WHICH WORKS OUT TO

$$10(0.5) - 6 = -1$$

YOU CAN CHECK IT USING THIS TABLE.

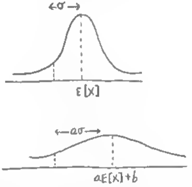
x	0	1
W	−6	4
$p(w)$	0.5	0.5

IN GENERAL, IT IS NOT HARD TO SHOW THAT

$$E[aX+b] = aE[X] + b$$

WHEN a AND b ARE ANY NUMBERS AND X IS ANY RANDOM VARIABLE. FOR THE VARIANCE, THERE'S ALSO A GENERAL RESULT:

$$\sigma^2(aX+b) = a^2\sigma^2(X)$$

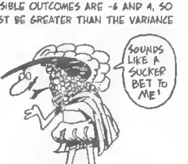


IN THE GAMBLING GAME ABOVE, THE POSSIBLE OUTCOMES ARE -6 AND 4, SO IT'S CLEAR THAT THE VARIANCE OF W MUST BE GREATER THAN THE VARIANCE OF X. IN FACT,

$$\sigma^2(W) = \sigma^2(10X+6) = 100\sigma^2(X) = 25$$

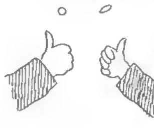
AND

$$\sigma(W) = 5$$



YOU CAN ALSO *ADD* TWO RANDOM VARIABLES TOGETHER. FOR INSTANCE, SUPPOSE WE TOSS A COIN *TWICE*. THE NUMBER OF HEADS ON BOTH TOSSES IS X_1+X_2, WHERE X_1 AND X_2 ARE THE RANDOM VARIABLES GIVING THE RESULTS OF THE FIRST AND SECOND TOSSES.

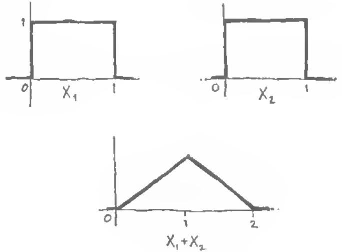
x_1+x_2	0	1	2
$p(x_1+x_2)$	.25	.5	.25



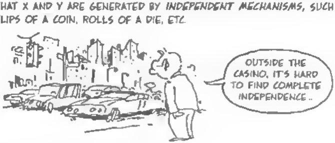
AGAIN, IT'S EASY TO SEE THAT

$$E[X_1+X_2] = E[X_1]+E[X_2]$$

(DON'T ASK ABOUT THE PROBABILITY DISTRIBUTION OF X_1+X_2, BECAUSE IT DEPENDS IN A COMPLICATED WAY ON THE TWO ORIGINAL DISTRIBUTIONS. FOR EXAMPLE, IF X_1 AND X_2 ARE BOTH THE SPINNING POINTER DISTRIBUTION, THE HISTOGRAMS ACT LIKE THIS:)



THE VARIANCE OF THE SUM OF RANDOM VARIABLES HAS A SIMPLE FORM IN THE SPECIAL CASE WHEN THE VARIABLES X AND Y ARE *INDEPENDENT.* THE TECHNICAL DEFINITION OF INDEPENDENCE IS BASED ON THE PROBABILITY PROPERTY P(A AND B) = P(A)P(B)... BUT FOR US, INDEPENDENCE JUST MEANS THAT X AND Y ARE GENERATED BY *INDEPENDENT MECHANISMS,* SUCH AS FLIPS OF A COIN, ROLLS OF A DIE, ETC.



WHEN X AND Y ARE INDEPENDENT, *THEIR VARIANCES ADD:*

$$\sigma^2(X+Y) = \sigma^2(X)+\sigma^2(Y)$$

IN THE CASE OF TWO COIN TOSSES,

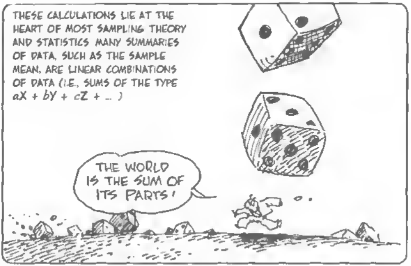
$$\begin{aligned}\sigma^2(X_1+X_2) &= \sigma^2(X_1)+\sigma^2(X_2)\\ &= .25+.25\\ &= .5\end{aligned}$$

ALL OF THIS CAN BE GENERALIZED TO THE SUM OF MANY RANDOM VARIABLES.

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

AND, WHEN THE X_i ARE ALL INDEPENDENT,

$$\sigma^2\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \sigma^2(X_i)$$

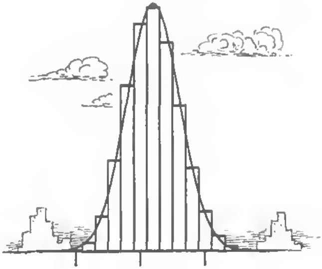


IN THE NEXT CHAPTER, WE WILL SEE TWO IMPORTANT EXAMPLES OF RANDOM VARIABLES: ONE, THE *BINOMIAL*, IS THE SUM OF MANY REPEATED INDEPENDENT RANDOM VARIABLES THE OTHER, THE *NORMAL*, IS A CONTINUOUS RANDOM VARIABLE THAT HAS A SURPRISING RELATIONSHIP TO THE BINOMIAL, AND ANY OTHER SUM OF INDEPENDENT RANDOM VARIABLES AS WELL.

♦Chapter 5♦

A TALE OF TWO DISTRIBUTIONS

NOW WE LOOK AT TWO IMPORTANT EXAMPLES OF RANDOM VARIABLES, ONE DISCRETE AND ONE CONTINUOUS.



WE BEGIN WITH THE DISCRETE ONE, CALLED THE *BINOMIAL* RANDOM VARIABLE SUPPOSE WE HAVE A RANDOM PROCESS WITH JUST *TWO POSSIBLE OUTCOMES:* A HEADS-OR-TAILS COIN TOSS, A WIN-OR-LOSE FOOTBALL GAME, A PASS-OR-FAIL AUTOMOTIVE SMOG INSPECTION. WE ARBITRARILY CALL ONE OF THESE OUTCOMES A *SUCCESS* AND THE OTHER A *FAILURE.*

WHAT WE DO IS TO REPEAT THIS EXPERIMENT.. WELL, REPEATEDLY. SUCH A REPEATABLE EXPERIMENT IS CALLED A

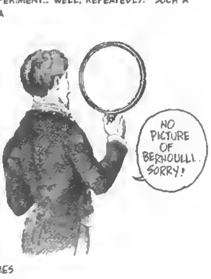
Bernoulli trial,

PROVIDED IT HAS THESE CRITICAL PROPERTIES.

1) THE RESULT OF EACH TRIAL MAY BE EITHER A SUCCESS OR A FAILURE

2) THE PROBABILITY p OF SUCCESS IS THE SAME IN EVERY TRIAL.

3) THE TRIALS ARE *INDEPENDENT:* THE OUTCOME OF ONE TRIAL HAS NO INFLUENCE ON LATER OUTCOMES



STARTING WITH A BERNOULLI TRIAL, WITH PROBABILITY OF SUCCESS p, LET'S BUILD A NEW RANDOM VARIABLE BY REPEATING THE BERNOULLI TRIAL.

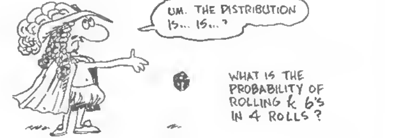
The binomial random variable

X IS THE *NUMBER OF SUCCESSES* IN n REPEATED BERNOULLI TRIALS WITH PROBABILITY p OF SUCCESS

AN EXAMPLE OF A BINOMIAL RANDOM VARIABLE IS THE NUMBER OF HEADS (SUCCESSES) IN TWO FLIPS OF A COIN. HERE $n=2$ AND $p=.5$

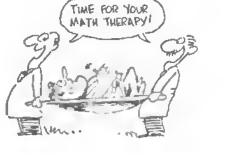
k = NUMBER OF SUCCESSES	0	1	2
$Pr(X=k)$	.25	.5	.25

ANOTHER EXAMPLE IS DE MERE'S FIRST GAMBLE: TOSSING A SINGLE DIE FOUR TIMES IN A ROW. SUCCESS MEANS ROLLING A 6. THE DISTRIBUTION IS:



IN GENERAL, WHAT'S THE PROBABILITY DISTRIBUTION OF THE BINOMIAL FOR *ANY* PROBABILITY p AND NUMBER OF TRIALS n? A PROBABILITY CALCULATION GIVES THE ANSWER. THE PROBABILITY OF OBTAINING k SUCCESSES IN n TRIALS, $Pr(X=k)$, IS

$$Pr(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$



HERE $\binom{n}{k}$, READ "n CHOOSE k," IS THE *BINOMIAL COEFFICIENT*. IT COUNTS ALL POSSIBLE WAYS OF GETTING k SUCCESSES IN n TRIALS. EACH INDIVIDUAL SEQUENCE OF k SUCCESSES AND $n-k$ FAILURES HAS PROBABILITY $p^k(1-p)^{n-k}$, BY THE MULTIPLICATION RULE. THERE ARE $\binom{n}{k}$ OF THESE SEQUENCES

THE FORMULA FOR $\binom{n}{k}$ IS

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

WHERE

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 1$$

AND $0!$ IS TAKEN TO BE 1. FOR INSTANCE, $\binom{4}{2}$, THE NUMBER OF POSSIBLE WAYS TO CHOOSE TWO LETTERS FROM A SET OF FOUR LETTERS, IS

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{4} = 6$$

ANOTHER VIEW OF THE BINOMIAL COEFFICIENTS IS IN *PASCAL'S TRIANGLE*. EACH ENTRY IS THE SUM OF THE TWO NUMBERS JUST ABOVE IT

$\binom{3}{1}$ $\binom{7}{2}$ $\binom{5}{3}$ $\binom{8}{0}$ $\binom{10}{7}$ $\binom{10}{10}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1

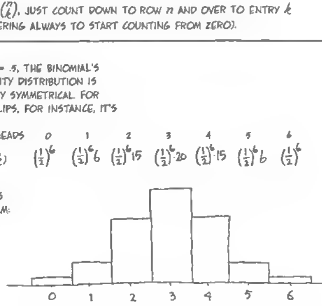
ETC.

TO FIND $\binom{n}{k}$, JUST COUNT DOWN TO ROW n AND OVER TO ENTRY k (REMEMBERING ALWAYS TO START COUNTING FROM ZERO).

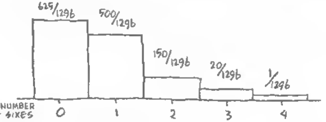
WHEN $p = .5$, THE BINOMIAL'S PROBABILITY DISTRIBUTION IS PERFECTLY SYMMETRICAL. FOR 6 COIN FLIPS, FOR INSTANCE, IT'S

k = # HEADS	0	1	2	3	4	5	6
$Pr(X=k)$	$\left(\frac{1}{2}\right)^6$	$\left(\frac{1}{2}\right)^6 6$	$\left(\frac{1}{2}\right)^6 15$	$\left(\frac{1}{2}\right)^6 \cdot 20$	$\left(\frac{1}{2}\right)^6 \cdot 15$	$\left(\frac{1}{2}\right)^6 6$	$\left(\frac{1}{2}\right)^6$

WITH THIS HISTOGRAM:



FOR DE MERE'S ROLL OF FOUR DICE, THE DISTRIBUTION IS MORE LOPSIDED:



THE *MEAN* AND *VARIANCE* OF THE BINOMIAL DISTRIBUTION ARE

$$\mu = np$$

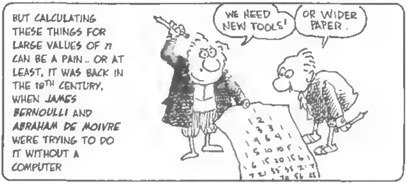
$$\sigma^2 = np(1-p)$$

NOTE THAT THE MEAN MAKES INTUITIVE SENSE. IN n BERNOULLI TRIALS, THE EXPECTED NUMBER OF SUCCESSES SHOULD BE np. THE VARIANCE FOLLOWS FROM THE FACT THAT THE BINOMIAL IS THE SUM OF n INDEPENDENT BERNOULLI TRIALS OF VARIANCE $p(1-p)$.

THE PARAMETERS OF THE BINOMIAL DISTRIBUTION ARE n AND p. THE DISTRIBUTION, MEAN, AND VARIANCE DEPEND *ONLY* ON THESE TWO NUMBERS. TABLES OF THE BINOMIAL DISTRIBUTION APPEAR IN MOST TEXTBOOKS AND COMPUTER PROGRAMS. HERE IS A TABLE FOR $n = 10$.

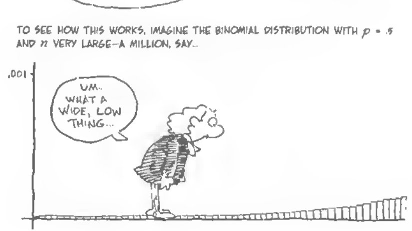
VALUES OF $Pr(X=k)$

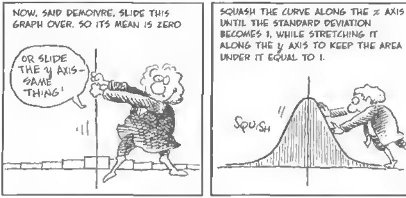
	k										
p	0	1	2	3	4	5	6	7	8	9	10
1	0.349	0.387	0.194	0.057	0.011	0.001	0.000	0.000	0.000	0.000	0.000
25	0.056	0.188	0.282	0.250	0.146	0.058	0.016	0.003	0.000	0.000	0.000
50	0.001	0.010	0.044	0.117	0.205	0.246	0.205	0.117	0.044	0.010	0.001
75	0.000	0.000	0.000	0.003	0.016	0.058	0.146	0.250	0.282	0.188	0.056
9	0.000	0.000	0.000	0.000	0.000	0.001	0.011	0.057	0.194	0.387	0.349



DEPLOYING A NEWLY INVENTED WEAPON, THE ***CALCULUS***, DE MOIVRE SHOWED THAN WHEN $p = .5$, THE BINOMIAL DISTRIBUTION WAS CLOSELY APPROXIMATED BY A ***CONTINUOUS DENSITY FUNCTION*** WHICH COULD BE DESCRIBED VERY SIMPLY.

TO SEE HOW THIS WORKS, IMAGINE THE BINOMIAL DISTRIBUTION WITH $p = .5$ AND n VERY LARGE—A MILLION, SAY..



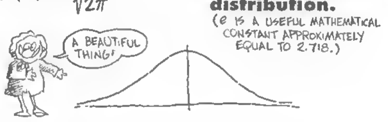


THE RESULT IS VERY CLOSE TO A *SMOOTH, SYMMETRICAL, BELL-SHAPED CURVE,* WHICH DEMOIVRE SHOWED WAS GIVEN BY THE SIMPLE FORMULA:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

THIS FUNCTION IS CALLED THE **standard normal distribution.**

(e IS A USEFUL MATHEMATICAL CONSTANT APPROXIMATELY EQUAL TO 2.718.)



(CONVINCE YOURSELF THAT THIS FUNCTION REALLY HAS A BELL-SHAPED GRAPH. FOR z FAR FROM ZERO, $f(z)$ IS VERY NEARLY ZERO—IT HAS A BIG DENOMINATOR; IT'S SYMMETRICAL, SINCE $f(z) = f(-z)$, AND IT HAS A MAXIMUM AT $z = 0$)

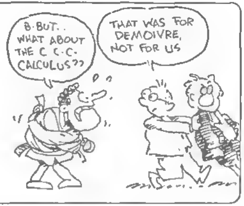
THE DISTRIBUTION IS CALLED THE *STANDARD* NORMAL BECAUSE ALL THAT SQUASHING AND STRETCHING WAS SPECIALLY ARRANGED TO GIVE IT THESE SIMPLE PROPERTIES, WHICH WE PRESENT WITHOUT PROOF.

$$\mu = 0$$

$$\sigma = 1$$

TO SUMMARIZE DE MOIVRE, IF YOU *"NORMALIZE"* THE *BINOMIAL* DISTRIBUTION WITH $p = 1/2$—I.E., CENTER IT ON ZERO AND MAKE ITS STANDARD DEVIATION = 1, THEN IT CLOSELY FITS THE *STANDARD NORMAL DISTRIBUTION*

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

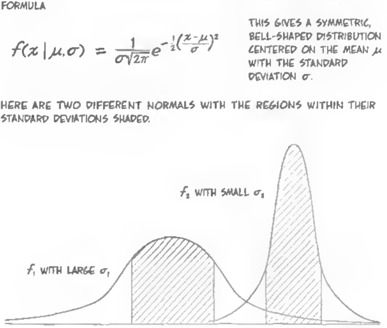


OTHER NORMALS, WITH DIFFERENT MEANS AND VARIANCES, ARE OBTAINED BY STRETCHING AND SLIDING THE STANDARD NORMAL. IN GENERAL, WE WRITE THE FORMULA

$$f(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

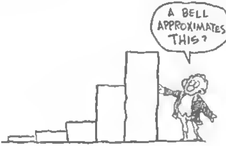
THIS GIVES A SYMMETRIC, BELL-SHAPED DISTRIBUTION CENTERED ON THE MEAN μ WITH THE STANDARD DEVIATION σ.

HERE ARE TWO DIFFERENT NORMALS WITH THE REGIONS WITHIN THEIR STANDARD DEVIATIONS SHADED.



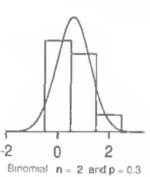
DE MOIVRE PROVED THAT THE STANDARD NORMAL FITS THE (NORMALIZED) BINOMIAL WITH $p = .5$. BUT, IN FACT, IT WORKS FOR *ANY VALUE OF* p.

GENERALLY: FOR ANY VALUE OF p, THE BINOMIAL DISTRIBUTION OF n TRIALS WITH PROBABILITY p IS APPROXIMATED BY THE NORMAL CURVE WITH $\mu = np$ AND $\sigma = np(1-p)$.

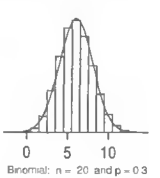


THIS IS ACTUALLY A LITTLE STRANGE. ALL NORMALS ARE SYMMETRICAL AND BELL SHAPED. BUT, AS WE SAW, BINOMIAL DISTRIBUTIONS ARE *NOT SYMMETRICAL* WHEN $p \neq .5$.

BUT IT TURNS OUT THAT AS n GETS LARGE, THE BINOMIAL'S ASYMMETRY IS OVERWHELMED, AS YOU SEE IN THIS EXAMPLE:



Binomial $n = 2$ and $p = 0.3$



Binomial: $n = 20$ and $p = 0.3$

IN FACT, DEMOIVRE'S DISCOVERY ABOUT THE BINOMIAL IS A SPECIAL CASE OF AN EVEN *MORE* GENERAL RESULT, WHICH HELPS EXPLAIN WHY THE NORMAL IS SO IMPORTANT AND WIDESPREAD IN NATURE. IT IS THIS:

"Fuzzy Central Limit Theorem":

DATA THAT ARE INFLUENCED BY ***MANY SMALL AND UNRELATED RANDOM EFFECTS*** ARE APPROXIMATELY ***NORMALLY DISTRIBUTED.***

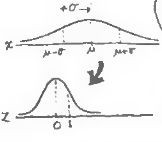
THIS EXPLAINS WHY THE NORMAL IS *EVERYWHERE:* STOCK MARKET FLUCTUATIONS, STUDENT WEIGHTS, YEARLY TEMPERATURE AVERAGES, S.A.T. SCORES: ALL ARE THE RESULT OF MANY DIFFERENT EFFECTS. FOR EXAMPLE, A STUDENT'S WEIGHT IS THE RESULT OF GENETICS, NUTRITION, ILLNESS, AND LAST NIGHT'S BEER PARTY. WHEN YOU PUT THEM ALL TOGETHER, YOU GET THE NORMAL! (REMEMBER, THE BINOMIAL IS THE RESULT OF n INDEPENDENT BERNOULLI TRIALS.)

NOW, BACK TO THE MATH...

THE z TRANSFORMATION

$$z = \frac{x-\mu}{\sigma}$$

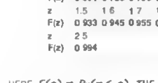
CHANGES A NORMAL RANDOM VARIABLE WITH MEAN μ AND STANDARD DEVIATION σ INTO A *STANDARD* NORMAL RANDOM VARIABLE WITH MEAN 0 AND STANDARD DEVIATION 1.



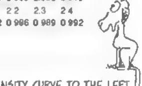
THEN ALL WE NEED TO FIND PROBABILITIES FOR *ANY* NORMAL DISTRIBUTION IS THE SINGLE TABLE FOR THE *STANDARD NORMAL* $F(z)$

z	-2.5	-2.4	-2.3	-2.2	-2.1	-2.0	-1.9	-1.8	-1.7	-1.6
F(z)	0.006	0.008	0.011	0.014	0.019	0.023	0.029	0.036	0.045	0.055
z	-1.5	-1.4	-1.3	-1.2	-1.1	-1.0	-0.9	-0.8	-0.7	-0.6
F(z)	0.067	0.081	0.097	0.115	0.136	0.159	0.184	0.212	0.242	0.274
z	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4
F(z)	0.309	0.345	0.382	0.421	0.460	0.500	0.540	0.579	0.618	0.655
z	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
F(z)	0.691	0.726	0.758	0.788	0.816	0.841	0.864	0.885	0.903	0.919
z	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
F(z)	0.933	0.945	0.955	0.964	0.971	0.977	0.982	0.986	0.989	0.992
z	2.5									
F(z)	0.994									

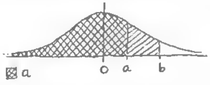
HERE $F(a) = Pr(z \leq a)$, THE AREA UNDER THE DENSITY CURVE TO THE LEFT OF $z = a$.



(WE CAN ALSO GRAPH THE CURVE $y = F(z)$, THE *CUMULATIVE PROBABILITY*. IT LOOKS LIKE THIS.)



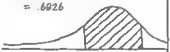
THE TABLE ALLOWS US TO FIND THE PROBABILITY OF Z BEING IN ANY INTERVAL $a \le z \le b$. IT IS JUST THE DIFFERENCE BETWEEN THE AREAS $F(b)$ AND $F(a)$.



$$Pr(a \le z \le b) = F(b) - F(a)$$

SO, FOR EXAMPLE,

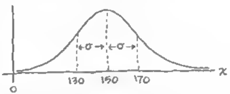
$$Pr(-1 < z < 1) = F(1) - F(-1)$$
$$= .8413 - .1587$$
$$= .6826$$



$$Pr(z \ge 2) = 1 - F(2)$$
$$= 1 - .9772$$
$$= .0228$$

USING THE SUBSTITUTION $z = \frac{x - \mu}{\sigma}$, WE CAN USE THE SAME TABLE TO FIND PROBABILITIES FOR OTHER NORMAL DISTRIBUTIONS

FOR EXAMPLE, SUPPOSE STUDENT WEIGHTS ARE NORMALLY DISTRIBUTED WITH A MEAN $\mu = 150$ POUNDS AND STANDARD DEVIATION $\sigma = 20$.



THEN WHAT'S THE PROBABILITY OF WEIGHING MORE THAN 170 POUNDS?

NOW IT'S "JUST" ALGEBRA.

$$Pr(X > 170) =$$
$$Pr\left(\frac{X - \mu}{\sigma} > \frac{170 - 150}{20}\right) =$$
$$Pr\left(Z > \frac{20}{20}\right) =$$
$$Pr(Z > 1)$$

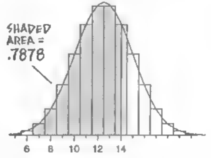
THAT'S $1 - F(1)$, WHICH WE CAN READ FROM THE TABLE AS $1 - .8413 = .1587$

A LITTLE LESS THAN ONE STUDENT IN SIX TIPS THE SCALES ABOVE 170 POUNDS.

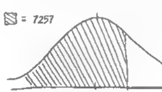
THE GENERAL RULE FOR COMPUTING NORMAL PROBABILITIES IS THEREFORE:

$$Pr(a \le X \le b) = F\left(\frac{b - \mu}{\sigma}\right) - F\left(\frac{a - \mu}{\sigma}\right)$$

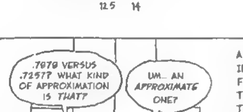
NOW BACK TO DE MOIVRE AND HIS BINOMIAL APPROXIMATION... LET'S LOOK AT A BINOMIAL DISTRIBUTION WITH $n = 25$ TRIALS AND $p = .5$ (25 COIN FLIPS, SAY). WE CAN COMPUTE (OR LOOK UP IN A TABLE) ANY PROBABILITY, FOR EXAMPLE, $Pr(X \le 14)$. IT IS **.7878** EXACTLY



NOW CALCULATE A NORMAL RANDOM VARIABLE X^* WITH THE SAME MEAN $\mu = np = (25)(.5) = 12.5$ AND STANDARD DEVIATION $\sigma = np(1-p) = 2.5$.



$$Pr(X^* \le 14) = Pr\left(Z \le \frac{14-12.5}{2.5}\right)$$
$$= Pr(Z \le .6)$$
$$= \mathbf{.7257}$$

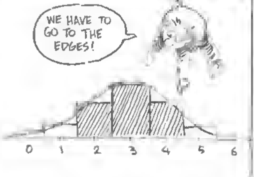


AH, BUT WE CAN DO BETTER! IF YOU LOOK CLOSELY AT THE FIRST HISTOGRAM, YOU SEE THE BARS ARE *CENTERED* ON THE NUMBERS. THIS MEANS $Pr(X^* \le 14)$ IS ACTUALLY THE AREA UNDER THE BARS LESS THAN $x = 14.5$ WE NEED TO ACCOUNT FOR THAT EXTRA .5, AND IN FACT,

$$Pr(X^* \le 14.5) = Pr(Z \le .9)$$
$$= \mathbf{.7881}$$

A VERY GOOD APPROXIMATION TO .7878 INDEED!

THAT LITTLE EXTRA .5 WE ADDED IS CALLED THE **continuity correction.** WE HAVE TO INCLUDE IT TO GET A GOOD CONTINUOUS APPROXIMATION TO OUR DISCRETE BINOMIAL RANDOM VARIABLE X. IT'S SUMMARIZED BY THIS ONE HIDEOUS FORMULA.



$$\Pr(a \le X \le b) \simeq \Pr\left(\frac{a-\frac{1}{2}-np}{\sqrt{np(1-p)}} \le Z \le \frac{b+\frac{1}{2}-np}{\sqrt{np(1-p)}}\right)$$

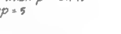
WHEN IS THIS APPROXIMATION "GOOD ENOUGH?" FOR STATISTICIANS, THE RULE OF THUMB IS: WHENEVER n IS BIG ENOUGH TO MAKE THE NUMBER OF EXPECTED SUCCESSES AND FAILURES BOTH GREATER THAN *FIVE:*

$$np \geq 5 \quad and \quad n(1-p) \geq 5$$

YOU CAN SEE FROM THESE HISTOGRAMS THAT THE FIT WHEN $p = 0.1$ IS MEDIOCRE OR WORSE UNTIL n REACHES 50, MAKING $np = 5$

$n=2,\ p=0.1$

$n=10,\ p=0.1$

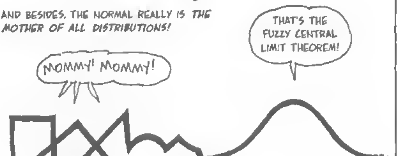


$n=50,\ p=0.1$

WHAT'S SO GREAT ABOUT THIS NORMAL APPROXIMATION? THE BINOMIAL DISTRIBUTION OCCURS COMMONLY IN NATURE, AND IT ISN'T HARD TO UNDERSTAND, BUT IT CAN BE TIRESOME TO CALCULATE.

THE NORMAL WHICH APPROXIMATES IT MAY BE LESS INTUITIVE, BUT IT'S VERY EASY TO USE. THE z-TRANSFORM CONVERTS *ANY* NORMAL TO THE *STANDARD NORMAL*, ALLOWING US TO READ PROBABILITIES STRAIGHT OUT OF A SINGLE NUMERICAL TABLE.

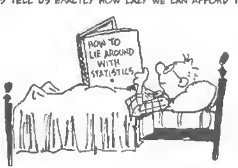
AND BESIDES, THE NORMAL REALLY IS *THE MOTHER OF ALL DISTRIBUTIONS!*



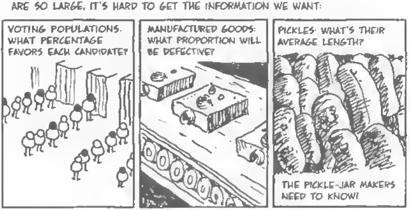
♦Chapter 6♦
SAMPLING

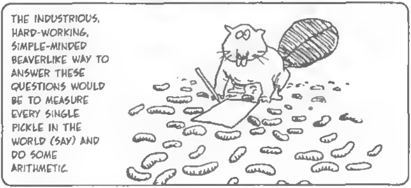
BY NOW, AFTER A STEADY DIET OF COINS, DICE, AND ABSTRACT IDEAS, YOU MAY BE WONDERING WHAT ALL THIS STATISTICAL EQUIPMENT WE'VE BEEN BUILDING HAS TO DO WITH THE *REAL WORLD.* WELL, NOW WE'RE FINALLY GOING TO FIND OUT...

IN THIS CHAPTER, WE BEGIN LOOKING AT THE *REAL* BUSINESS OF STATISTICS, WHICH IS, AFTER ALL, TO SAVE PEOPLE *TIME* AND *MONEY.* PEOPLE HATE TO WASTE TIME DOING *UNNECESSARY WORK,* AND ONE THING STATISTICS CAN DO IS TELL US EXACTLY HOW LAZY WE CAN AFFORD TO BE.



THE PROBLEM WITH THE WORLD IS THAT THE COLLECTIONS OF STUFF IN IT ARE SO LARGE, IT'S HARD TO GET THE INFORMATION WE WANT:





BUT WE AREN'T BEAVERS—WE'RE ***STATISTICIANS!*** WE'RE LOOKING FOR THE ***EASY*** WAY OUT...

OUR METHOD IS TO TAKE A ***SAMPLE...*** A RELATIVELY SMALL SUBSET OF THE TOTAL POPULATION, THE WAY POLLSTERS DO AT ELECTION TIME.

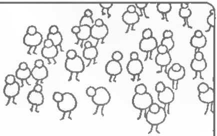
AN OBVIOUS QUESTION IS HOW BIG A SAMPLE DO WE HAVE TO TAKE TO GET MEANINGFUL RESULTS?

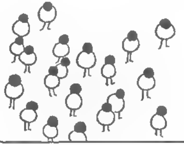
AND THE ANSWER, WHICH YOU SHOULD INSCRIBE IN YOUR BRAIN FOREVERMORE, WILL TURN OUT TO BE: IF n IS THE NUMBER OF ITEMS IN THE SAMPLE, THEN EVERYTHING IS GOVERNED BY

$$\frac{1}{\sqrt{n}}.$$

SAMPLING DESIGN

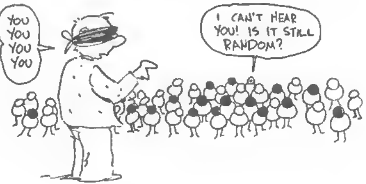
BEFORE DOING THE NUMBERS, WE SHOULD POINT OUT THAT THE *QUALITY* OF THE SAMPLE IS AS IMPORTANT AS ITS *SIZE*. HOW DO WE ASSURE OURSELVES THAT WE'RE CHOOSING A REPRESENTATIVE SAMPLE?





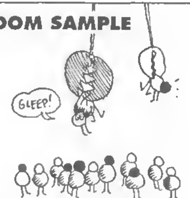
THE *SELECTION PROCESS ITSELF* IS CRITICAL. FOR EXAMPLE, A VOTER SURVEY THAT SYSTEMATICALLY EXCLUDED BLACK PEOPLE WOULD BE WORTHLESS, AND THERE ARE A HOST OF OTHER WAYS TO RUIN, OR BIAS, A SAMPLE.

NOT TO PROLONG THE MYSTERY, THE WAY TO GET STATISTICALLY DEPENDABLE RESULTS IS TO CHOOSE THE SAMPLE AT **random.**



THE **SIMPLE** RANDOM SAMPLE

SUPPOSE WE HAVE A LARGE POPULATION OF OBJECTS AND A PROCEDURE FOR SELECTING n OF THEM. IF THE PROCEDURE ENSURES THAT ***ALL POSSIBLE SAMPLES*** OF n OBJECTS ARE ***EQUALLY LIKELY,*** THEN WE CALL THE PROCEDURE A **simple random sample.**

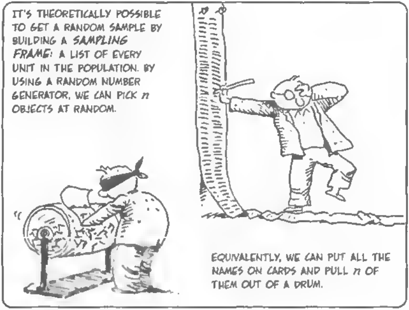


THE SIMPLE RANDOM SAMPLE HAS TWO PROPERTIES THAT MAKE IT THE STANDARD AGAINST WHICH WE MEASURE ALL OTHER METHODS

1) UNBIASED: EACH UNIT HAS THE SAME CHANCE OF BEING CHOSEN.

2) INDEPENDENCE: SELECTION OF ONE UNIT HAS NO INFLUENCE ON THE SELECTION OF OTHER UNITS.

UNFORTUNATELY, IN THE REAL WORLD, COMPLETELY UNBIASED, INDEPENDENT SAMPLES ARE HARD TO FIND. FOR INSTANCE, SURVEYING VOTERS BY RANDOMLY DIALING TELEPHONE NUMBERS IS BIASED: IT IGNORES VOTERS WITHOUT A TELEPHONE AND OVERSAMPLES PEOPLE WITH MORE THAN ONE NUMBER.



BUT THIS IS NOT ALWAYS EASY. MAKING THE FRAME MAY BE PROHIBITIVELY COSTLY, CONTROVERSIAL, OR EVEN IMPOSSIBLE. FOR EXAMPLE, AN E.P.A. WATER QUALITY STUDY NEEDED A SAMPLING FRAME OF LAKES IN THE U.S., SO THEN SOMEBODY HAS TO DECIDE:

ARE THERE OTHER WAYS TO SAMPLE THAT ARE MORE *EFFICIENT* AND *COST-EFFECTIVE* THAN A SIMPLE RANDOM SAMPLE? YES—IF YOU ALREADY KNOW SOMETHING ABOUT THE POPULATION. FOR INSTANCE...

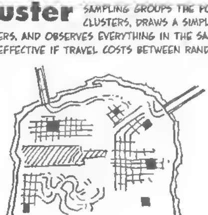
Stratified

SAMPLING. DIVIDE THE POPULATION UNITS INTO HOMOGENEOUS GROUPS *(STRATA)* AND DRAW A SIMPLE RANDOM SAMPLE FROM EACH GROUP.

FOR EXAMPLE, THE POPULATION OF ALL *PICKLES* CAN BE STRATIFIED BY *TYPE OF PICKLE*. WITHIN EACH TYPE OR STRATUM, THE SIZE SHOULD BE LESS VARIABLE.

Cluster

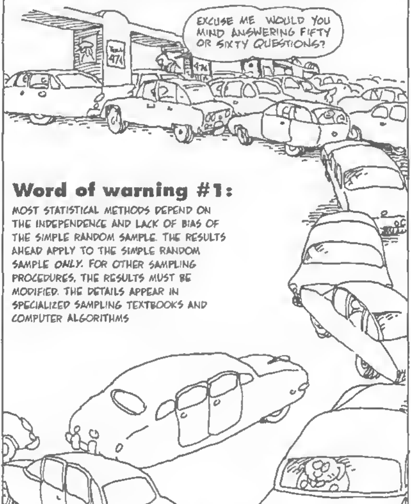
SAMPLING GROUPS THE POPULATION INTO SMALL CLUSTERS, DRAWS A SIMPLE RANDOM SAMPLE OF CLUSTERS, AND OBSERVES EVERYTHING IN THE SAMPLED CLUSTERS. THIS CAN BE COST-EFFECTIVE IF TRAVEL COSTS BETWEEN RANDOMLY SAMPLED UNITS IS HIGH.



AN EXAMPLE IS A CITY HOUSING SURVEY WHICH DIVIDES A CITY INTO BLOCKS, RANDOMLY SAMPLES THE BLOCKS, AND LOOKS AT EVERY HOUSING UNIT IN EACH SAMPLED BLOCK.

Systematic

SAMPLING STARTS WITH A RANDOMLY CHOSEN UNIT AND THEN SELECTS EVERY k^{TH} UNIT THEREAFTER. FOR INSTANCE, A *HIGHWAY TRAFFIC STUDY* MIGHT CHECK EVERY HUNDREDTH CAR AT A TOLL BOOTH. THIS PLAN IS EASY TO IMPLEMENT AND CAN BE MORE EFFICIENT IF TRAFFIC PATTERNS VARY SMOOTHLY OVER TIME.



Word of warning #1:

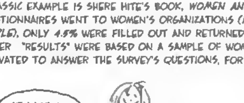
MOST STATISTICAL METHODS DEPEND ON THE INDEPENDENCE AND LACK OF BIAS OF THE SIMPLE RANDOM SAMPLE. THE RESULTS AHEAD APPLY TO THE SIMPLE RANDOM SAMPLE *ONLY*. FOR OTHER SAMPLING PROCEDURES, THE RESULTS MUST BE MODIFIED. THE DETAILS APPEAR IN SPECIALIZED SAMPLING TEXTBOOKS AND COMPUTER ALGORITHMS.

Word of warning #2:

WITHOUT RANDOMIZED DESIGN, THERE CAN BE NO DEPENDABLE STATISTICAL ANALYSIS, NO MATTER HOW IT IS MODIFIED. THE BEAUTY OF RANDOM SAMPLING IS THAT IT "STATISTICALLY GUARANTEES" THE ACCURACY OF THE SURVEY.

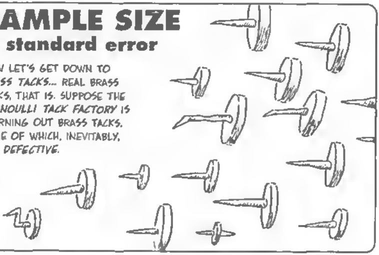
A COMMONLY USED METHOD IS ESPECIALLY PRONE TO BIAS IT'S CALLED AN **opportunity** SAMPLE. AVOIDING ALL THE BOTHER OF DESIGNING A PROCEDURE, THE OPPORTUNITY SAMPLER JUST GRABS THE FIRST n POPULATION UNITS TO COME ALONG

A CLASSIC EXAMPLE IS SHERE HITE'S BOOK, *WOMEN AND LOVE*. 100,000 QUESTIONNAIRES WENT TO WOMEN'S ORGANIZATIONS (AN *OPPORTUNITY SAMPLE*), ONLY 4.5% WERE FILLED OUT AND RETURNED (*RESPONSE BIAS*). SO HER "RESULTS" WERE BASED ON A SAMPLE OF WOMEN WHO WERE HIGHLY MOTIVATED TO ANSWER THE SURVEY'S QUESTIONS, FOR WHATEVER REASON.

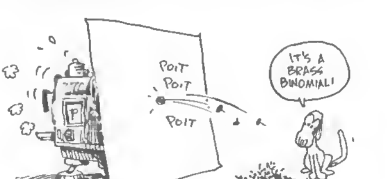


SAMPLE SIZE
& standard error

NOW LET'S GET DOWN TO *BRASS TACKS*... REAL BRASS TACKS, THAT IS. SUPPOSE THE *BERNOULLI TACK FACTORY* IS CHURNING OUT BRASS TACKS, SOME OF WHICH, INEVITABLY, ARE *DEFECTIVE*.



THE ASTUTE READER WILL RECOGNIZE THIS AS A *BERNOULLI SYSTEM*: EACH NEW TACK IS THE OUTCOME OF A BERNOULLI TRIAL WITH SOME PROBABILITY p OF SUCCESS (I.E., BEING DEFECT-FREE) AND PROBABILITY $1-p$ OF FAILURE (I.E., BEING DEFECTIVE).



WE THINK OF THIS SITUATION AS IF THERE WERE A *HIDDEN BUT REAL "BERNOULLI MACHINE"* WHOSE PROBABILITY p GOVERNS THE OUTCOMES WE OBSERVE IN THE SO-CALLED "REAL WORLD."

SINCE THE BERNOULLI MACHINE IS INVISIBLE, WE DON'T KNOW WHAT p IS, BUT WE'D LIKE TO FIND OUT. SO WE TAKE A *RANDOM SAMPLE* OF n TACKS, AND FIND THAT x OF THEM ARE O.K.

NOW THE PROPORTION OF SUCCESSES IN THE SAMPLE SHOULD BE SOMEWHERE AROUND p... SO WE CALL IT $\hat{p}$, PRONOUNCED "P-HAT"

$$\hat{p} = \frac{x}{n}$$

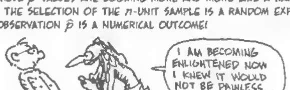
$\hat{p}$ IS THE NUMBER OF SUCCESSES x IN THE SAMPLE, DIVIDED BY THE SAMPLE SIZE n. FOR EXAMPLE, IF p WAS .85, AND WE SAMPLED $n = 1000$ TACKS, MAYBE WE FOUND $x = 832$ GOOD ONES, MAKING $\hat{p} = .832$.

WE ASK: HOW GOOD IS THIS ESTIMATE?

AND WE ANSWER WITH ANOTHER QUESTION: WHAT DOES THE FIRST QUESTION MEAN?

WE CAN'T KNOW THE PRECISE DIFFERENCE BETWEEN $\hat{p}$ AND p, BECAUSE WE DON'T KNOW THE VALUE OF p. THE *REAL* QUESTION IS THIS: IF WE TOOK *MANY SAMPLES OF 1000 TACKS* AND OBSERVED $\hat{p}$ FOR EACH SAMPLE, HOW WOULD THOSE VALUES OF $\hat{p}$ BE *DISTRIBUTED* AROUND p?

IN FACT, THESE $\hat{p}$ VALUES ARE LOOKING MORE AND MORE LIKE A *RANDOM VARIABLE*: THE SELECTION OF THE n-UNIT SAMPLE IS A RANDOM EXPERIMENT, AND THE OBSERVATION $\hat{p}$ IS A NUMERICAL OUTCOME!



TO BE PRECISE, IF X IS THE NUMBER OF SUCCESSES IN THE SAMPLE, THEN X IS NOTHING BUT OUR OLD FRIEND THE BINOMIAL RANDOM VARIABLE (n TRIALS, PROBABILITY p)... AND WE DEFINE THE *OBSERVED PROPORTION* TO BE THE RANDOM VARIABLE

$$\hat{P} = \frac{X}{n}$$

KNOWING ALL ABOUT X, WE QUICKLY CONCLUDE A FEW FACTS ABOUT $\hat{P}$:

1) THE MEAN OF $\hat{P}$ IS $E[\hat{P}] = p$

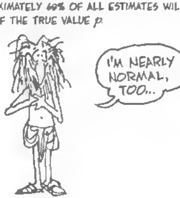
2) THE STANDARD DEVIATION OF $\hat{P}$ IS

$$\sigma(\hat{P}) = \frac{\sqrt{p(1-p)}}{\sqrt{n}}$$

3) FOR LARGE n, $\hat{P}$ IS APPROXIMATELY NORMAL.

AND THERE YOU HAVE IT ALL! THE OBSERVED VALUES OF $\hat{P}$ WILL BE CENTERED ON p (NOT SURPRISINGLY), AND THEIR STANDARD DEVIATION, OR SPREAD, IS PROPORTIONAL TO THAT MAGIC NUMBER WE MENTIONED AT THE BEGINNING OF THE CHAPTER:

AND, SINCE $\hat{P}$ IS NEARLY NORMAL, WE CAN USE OUR RULE OF THUMB TO CONCLUDE THAT APPROXIMATELY 68% OF ALL ESTIMATES WILL FALL WITHIN ONE STANDARD DEVIATION OF THE TRUE VALUE p.



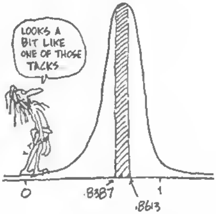
GOING BACK TO THE TACKS, WITH $n = 1000$ AND $p = .85$, WE GET A STANDARD DEVIATION OF

$$\sigma(\hat{P}) = \sqrt{\frac{(.85)(.15)}{1000}}$$

$$= .0113$$

SO WE EXPECT ABOUT 68% OF OUR ESTIMATES TO FALL IN THE NARROW INTERVAL

$$.8387 \le \hat{p} \le .8613$$



THE STANDARD DEVIATION OF $\hat{P}$ IS A MEASURE OF THE **sampling error.**

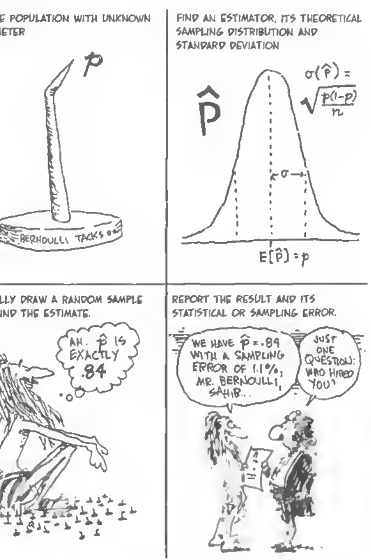
AS WE'VE SEEN, FOR THE BINOMIAL $\hat{P}$, THIS SAMPLING ERROR IS INVERSELY PROPORTIONAL TO $\sqrt{n}$. INCREASING THE SAMPLE SIZE BY A FACTOR OF 4 REDUCES THE SPREAD $\sigma(\hat{P})$ BY A FACTOR OF 2.

SAMPLE SIZES FOR TACKS, $p = 0.85$

n	1	4	16	25	100	10,000
$\sqrt{n}$	1	2	4	5	10	100
$\sigma(\hat{p})$	.357	.1785	.089	.071	.0357	.0036

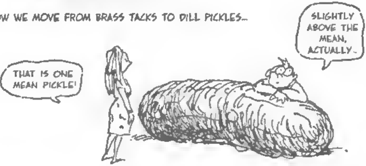
LINGUISTIC NOTE: AN *ESTIMATE* IS A SINGLE MEASURE OR OBSERVATION. AN *ESTIMATOR* IS A RULE FOR GETTING ESTIMATES. IN THIS CASE, THE ESTIMATOR IS THE RANDOM VARIABLE $\hat{P} = \frac{X}{n}$.

MOST OF STATISTICS INVOLVES THE 4-STEP PROCESS WE'VE JUST WALKED THROUGH:



Sampling Distribution of the MEAN

NOW WE MOVE FROM BRASS TACKS TO DILL PICKLES...



JAR MANUFACTURERS WOULD LIKE TO KNOW THE *AVERAGE LENGTH* OF A PICKLE WITHOUT EXAMINING EVERY CUCUMBER IN CALIFORNIA. THEY RANDOMLY SELECT n PICKLES AND MEASURE THEIR LENGTHS $x_1, x_2, \ldots, x_n$.

BY NOW YOU MAY BE USED TO THE IDEA THAT EACH X_i IS A *RANDOM VARIABLE:* THE NUMERICAL OUTCOME OF A RANDOM EXPERIMENT.

IF μ IS THE (UNKNOWN) MEAN PICKLE LENGTH, AND σ IS THE STANDARD DEVIATION OF THE *PICKLE LENGTH DISTRIBUTION*, THEN

$$E[X_i] = \mu$$
$$\sigma(X_i) = \sigma$$

FOR EVERY i (BECAUSE x_i COULD HAVE BEEN THE LENGTH OF ANY PICKLE).

NOW WE LOOK AT THE SAMPLE MEAN: THE AVERAGE LENGTH OF THE SELECTED PICKLES IT'S A NEW RANDOM VARIABLE GIVEN BY:

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

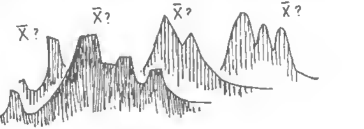
AS BEFORE, WE'D LIKE TO KNOW "HOW CLOSE" THIS IS TO μ, MEANING, IF THIS SAMPLING WERE DONE MANY TIMES, WHAT'S THE DISTRIBUTION OF $\bar{X}$? BECAUSE WE KNOW ABOUT $X_1, X_2, \dots$, AND X_n, WE ALSO KNOW THAT

$$E[\bar{X}] = \mu$$

$$\sigma(\bar{X}) = \sigma/\sqrt{n}$$

ONCE AGAIN, WE SEE THE MAGIC DENOMINATOR! THE SPREAD OF OBSERVED SAMPLE MEANS GOES AS $\frac{1}{\sqrt{n}}$.

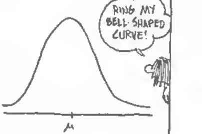
BUT WE DON'T KNOW THE SHAPE OF $\bar{X}$'S DISTRIBUTION. THE SAMPLE *PROBABILITY DISTRIBUTION* $\hat{p}$ WAS ALMOST NORMAL, BECAUSE IT WAS BASED ON A BINOMIAL RANDOM VARIABLE. BUT WHAT ABOUT $\bar{X}$, THE SAMPLE *MEAN* ESTIMATOR???



IT TURNS OUT THAT $\bar{X}$ IS ALSO APPROXIMATELY NORMAL! THIS FAMOUS RESULT IS CALLED THE

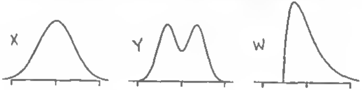
CENTRAL LIMIT THEOREM

IT SAYS: IF ONE TAKES RANDOM SAMPLES OF SIZE n FROM A POPULATION OF MEAN μ AND STANDARD DEVIATION σ, THEN, AS n GETS LARGE, $\bar{X}$ APPROACHES THE ***NORMAL DISTRIBUTION*** WITH MEAN μ AND STANDARD DEVIATION $\sigma/\sqrt{n}$. THEN

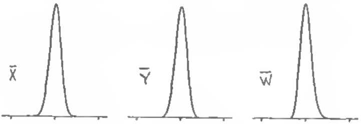


$$Pr(a \le \bar{X} \le b) = Pr\left(\frac{a-\mu}{\sigma/\sqrt{n}} \le Z \le \frac{b-\mu}{\sigma/\sqrt{n}}\right)$$

WHAT IS REMARKABLE ABOUT THIS? IT SAYS THAT REGARDLESS OF THE SHAPE OF THE ORIGINAL DISTRIBUTION (IN THIS CASE, OF PICKLE LENGTHS), THE TAKING OF ***AVERAGES*** RESULTS IN A *NORMAL*. TO FIND THE DISTRIBUTION OF $\bar{X}$, WE NEED KNOW *ONLY* THE POPULATION MEAN AND STANDARD DEVIATION.



THE THREE PROBABILITY DENSITIES ABOVE ALL HAVE THE SAME MEAN AND STANDARD DEVIATION. DESPITE THEIR DIFFERENT SHAPES, WHEN $n=10$, THE SAMPLING DISTRIBUTIONS OF THE MEAN, $\bar{X}$, ARE NEARLY IDENTICAL.



The t-distribution

AMAZING AS THE CENTRAL LIMIT THEOREM IS, IT HAS AT LEAST TWO PROBLEMS.

ONE: IT DEPENDS ON A LARGE SAMPLE SIZE

TWO: TO USE IT, WE NEED TO KNOW σ, THE STANDARD DEVIATION.

BUT SAMPLE SIZES ARE OFTEN SMALL, AND σ IS USUALLY UNKNOWN. CERTAINLY, IN THE CASE OF THE PICKLES, WE HAVE *NO IDEA* HOW WIDELY THEIR LENGTHS VARY AROUND THE AVERAGE.

WHAT WE CAN DO IN THIS CASE IS TO *ESTIMATE* σ BY TAKING THE *STANDARD DEVIATION OF THE SAMPLE*, WHICH, YOU'LL RECALL, IS GIVEN BY THE FORMULA

$$s = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

THEN, IN PLACE OF THE RANDOM VARIABLE

$$z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$

WE SUBSTITUTE s FOR σ, AND DEFINING A *NEW RANDOM VARIABLE* t BY

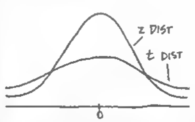
$$t = \frac{\bar{X}-\mu}{s/\sqrt{n}}$$

YOU CAN THINK OF THE RANDOM VARIABLE t AS *THE BEST WE CAN DO UNDER THE CIRCUMSTANCES*. ITS DISTRIBUTION IS CALLED *STUDENT'S* t, BECAUSE ITS INVENTOR, *WILLIAM GOSSET*, PUBLISHED UNDER THE PSEUDONYM "STUDENT"

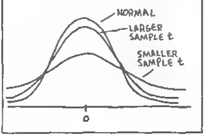
(GOSSET WAS EMPLOYED BY THE *GUINNESS BREWERY*, WHICH REQUIRED HIM TO USE A PSEUDONYM, FOR SOME REASON.)

MAKING THE ASSUMPTION THAT THE *ORIGINAL POPULATION DISTRIBUTION WAS NORMAL*, OR NEARLY NORMAL, "STUDENT" WAS ABLE TO CONCLUDE:

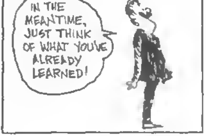
t IS MORE SPREAD OUT THAN Z. IT'S "FLATTER" THAN NORMAL. THIS IS BECAUSE THE USE OF s INTRODUCES MORE UNCERTAINTY, MAKING t "SLOPPIER" THAN Z.



THE AMOUNT OF SPREAD DEPENDS ON THE *SAMPLE SIZE*. THE GREATER THE SAMPLE SIZE, THE MORE CONFIDENT WE CAN BE THAT s IS NEAR σ, AND THE CLOSER t GETS TO z, THE NORMAL.



GOSSET WAS ABLE TO COMPUTE TABLES OF t FOR VARIOUS SAMPLE SIZES, WHICH WE WILL SEE HOW TO USE IN THE FOLLOWING CHAPTER.



IN THIS CHAPTER, WE CONSIDERED A CENTRAL PROBLEM OF *REAL-WORLD STATISTICS:* HOW TO SELECT A *SAMPLE* FROM A LARGE POPULATION SO THAT STATISTICAL ANALYSIS CAN BE VALID. BESIDES THE "GOLD STANDARD" OF THE SIMPLE RANDOM SAMPLE, WE ALSO DESCRIBED SOME OTHER SAMPLING SCHEMES THAT ARE USED IN THE INTERESTS OF EFFICIENCY, COST, AND PRACTICALITY.

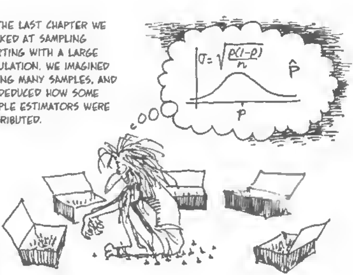
THEN, ASSUMING A SIMPLE RANDOM SAMPLE, WE CONSIDERED HOW VARIOUS SAMPLE STATISTICS WERE *DISTRIBUTED*. THAT IS, WE REGARDED THE ACT OF TAKING THE SAMPLE AS A *RANDOM EXPERIMENT*, SO THAT ITS STATISTICS BECAME *RANDOM VARIABLES*.

WE FOUND THAT SAMPLE *PROPORTIONS* $\hat{p}$ WERE APPROXIMATELY NORMALLY DISTRIBUTED, WHILE THE DISTRIBUTION OF THE SAMPLE *MEAN* $\bar{X}$ DEPENDED ON THE SAMPLE SIZE: FOR LARGE SAMPLES, THE DISTRIBUTION WAS APPROXIMATELY NORMAL, WHILE FOR SMALL SAMPLES, WE USE THE STUDENT'S t DISTRIBUTION.

IN THE NEXT TWO CHAPTERS, WE LOOK AT HOW TO USE THESE DISTRIBUTIONS TO MAKE *STATISTICAL INFERENCES:* GIVEN A SINGLE OBSERVATION, LIKE A POLITICAL POLL, HOW DO WE USE OUR KNOWLEDGE OF $\hat{p}$ AND $\bar{X}$ TO EVALUATE IT?

•Chapter 7•
CONFIDENCE INTERVALS

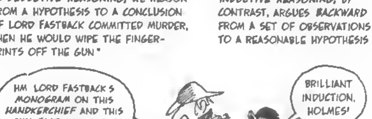
IN THE LAST CHAPTER WE LOOKED AT SAMPLING STARTING WITH A LARGE POPULATION. WE IMAGINED TAKING MANY SAMPLES, AND WE DEDUCED HOW SOME SAMPLE ESTIMATORS WERE DISTRIBUTED.



IN THIS CHAPTER, WE DO THE REVERSE: GIVEN *ONE* SAMPLE, WE ASK THE QUESTION, WHAT WAS THE RANDOM SYSTEM THAT GENERATED ITS STATISTICS?

IN *DEDUCTIVE REASONING,* WE REASON FROM A HYPOTHESIS TO A CONCLUSION "IF LORD FASTBACK COMMITTED MURDER, THEN HE WOULD WIPE THE FINGER-PRINTS OFF THE GUN."

INDUCTIVE REASONING, BY CONTRAST, ARGUES *BACKWARD* FROM A SET OF OBSERVATIONS TO A REASONABLE HYPOTHESIS



IN MANY WAYS, SCIENCE, INCLUDING STATISTICS, IS LIKE DETECTIVE WORK. BEGINNING WITH A SET OF OBSERVATIONS, WE ASK WHAT CAN BE SAID ABOUT THE SYSTEMS THAT GENERATED THEM.

ESTIMATING
CONFIDENCE INTERVALS

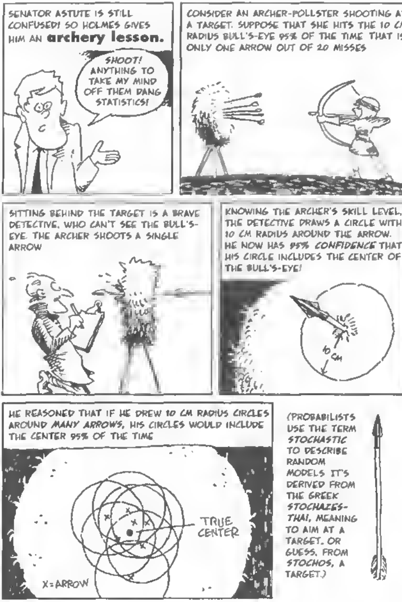
IS ONE OF THE MOST EFFECTIVE FORMS OF STATISTICAL INFERENCE, AND ONE YOU SEE EVERY DAY BEFORE *ELECTION TIME...*

IN A RECENT ELECTION SOMEWHERE, INCUMBENT SENATOR *ASTUTE* (ACCENT ON THE LAST SYLLABLE, PLEASE!) COMMISSIONED A POLL BY *BETTER HOLMES RESEARCH.* POLLSTER HOLMES DRAWS A SIMPLE RANDOM SAMPLE OF 1000 VOTERS AND ASKS THEM WHAT THEY THINK OF ASTUTE.

AFTER CENSORING THE REMARKS OF A FEW GRUMPY OUTLIERS, HOLMES FINDS THAT ***550*** VOTERS FAVOR HIS CLIENT, SENATOR ASTUTE.

THIS IS THE SINGLE OBSERVATION.


YOU ONLY ASKED A THOUSAND PEOPLE?! BUT THERE ARE A *MILLION* VOTERS IN THIS STATE!!!
I DON'T CARE IF THERE ARE A MILLION OR A BILLION.
I MADE A SIMPLE RANDOM SAMPLE! THIS MEANS I CAN OFFER YOU AN *IRONCLAD GUARANTEE!*
THIS GUARANTEE. IT'S AN ABSOLUTE, MONEY-BACK, TYPE GUARANTEE, RIGHT?
EH?
UM... AH... WELL... IT'S *STATISTICAL*, ACTUALLY...
YES. I CAN SAY WITH 95% *CONFIDENCE* THAT THE TRUE POPULATION PROPORTION, p, IS *BETWEEN .519 AND .581!*
95% CERTAIN??!! THUNDER!! WHAT DO YOU SUPPOSE WOULD HAPPEN IF I RAN ON A PLATFORM OF *95% HONESTY??*
I DON'T KNOW. IT'S NEVER BEEN TRIED...
AFTER ASTUTE CALMS DOWN, HOLMES EXPLAINS WHAT HE MEANS BY *95% CONFIDENCE*: HE KNOWS THAT HIS ESTIMATION PROCEDURE HAS A *95% PROBABILITY* OF PRODUCING AN INTERVAL CONTAINING p, I.E., IN HIS MANY YEARS OF POLLING, p HAS FALLEN WITHIN THE CONFIDENCE INTERVAL AROUND THE OBSERVED VALUE, $\hat{p}$, 95% OF THE TIME.

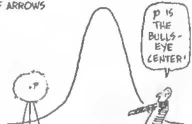

SENATOR ASTUTE IS STILL CONFUSED! SO HOLMES GIVES HIM AN **archery lesson.**
SHOOT! ANYTHING TO TAKE MY MIND OFF THEM DANG STATISTICS!
CONSIDER AN ARCHER-POLLSTER SHOOTING AT A TARGET. SUPPOSE THAT SHE HITS THE 10 CM RADIUS BULL'S-EYE 95% OF THE TIME THAT IS, ONLY ONE ARROW OUT OF 20 MISSES
SITTING BEHIND THE TARGET IS A BRAVE DETECTIVE, WHO CAN'T SEE THE BULL'S-EYE. THE ARCHER SHOOTS A SINGLE ARROW
KNOWING THE ARCHER'S SKILL LEVEL, THE DETECTIVE DRAWS A CIRCLE WITH 10 CM RADIUS AROUND THE ARROW. HE NOW HAS *95% CONFIDENCE* THAT HIS CIRCLE INCLUDES THE CENTER OF THE BULL'S-EYE!
10 CM
HE REASONED THAT IF HE DREW 10 CM RADIUS CIRCLES AROUND *MANY ARROWS*, HIS CIRCLES WOULD INCLUDE THE CENTER 95% OF THE TIME
TRUE CENTER
X=ARROW
(PROBABILISTS USE THE TERM *STOCHASTIC* TO DESCRIBE RANDOM MODELS. IT'S DERIVED FROM THE GREEK *STOCHAZESTHAI*, MEANING TO AIM AT A TARGET, OR GUESS, FROM *STOCHOS*, A TARGET.)

HOLMES NOW TRANSLATES THE ARCHERY LESSON INTO THE LANGUAGE WE DEVELOPED LAST CHAPTER.

Step One: SHOOT A LOT OF ARROWS

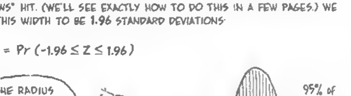
A PROBABILITY CALCULATION FINDS THE WIDTH OF THE "BULL'S-EYE." THE ESTIMATES $\hat{p}$ ARE OUR ARROWS. WE SAW THAT THE SAMPLING DISTRIBUTION OF $\hat{p}$ IS NEARLY *NORMAL* WITH MEAN p AND STANDARD DEVIATION

$$\sigma(\hat{p}) = \frac{\sqrt{p(1-p)}}{\sqrt{n}}$$



SINCE THE CURVE IS NORMAL, WE USE THE Z-TRANSFORM AND A STANDARD TABLE TO FIND THE WIDTH OF THE INTERVAL WITHIN WHICH 95% OF THE "ARROWS" HIT. (WE'LL SEE EXACTLY HOW TO DO THIS IN A FEW PAGES.) WE FIND THIS WIDTH TO BE 1.96 STANDARD DEVIATIONS.

$$.95 = Pr\,(-1.96 \le Z \le 1.96)$$



NOW WE DO SOME ALGEBRA BY DEFINITION OF THE Z-TRANSFORM.

$$.95 \simeq \Pr\left(-1.96 \le \frac{\hat{p}-p}{\sigma(p)} \le 1.96\right)$$

WHICH BECOMES

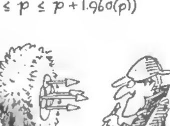
$$.95 \simeq \Pr\left(p-1.96\sigma(p) \le \hat{p} \le p+1.96\sigma(p)\right)$$

WHICH IS JUST ANOTHER WAY OF SAYING THAT 95% OF THE $\hat{p}$ "ARROWS" LAND BETWEEN $p - 1.96\sigma(p)$ AND $p + 1.96\sigma(p)$.

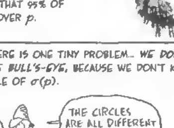
NOW WE'RE IN A POSITION TO VIEW THE TARGET FROM BEHIND! ONE MORE TURN OF THE ALGEBRA CRANK MAKES IT

$$.95 \simeq \Pr\left(\hat{p}-1.96\sigma(p) \le p \le \hat{p}+1.96\sigma(p)\right)$$

HERE WE ARE DRAWING CIRCLES AROUND A LOT OF ARROWS (I.E., MAKING INTERVALS AROUND $\hat{p}$) AND SAYING THAT 95% OF THEM COVER p.



BUT THERE IS ONE TINY PROBLEM... ***WE DON'T ACTUALLY KNOW THE SIZE OF THE BULL'S-EYE,*** BECAUSE WE DON'T KNOW p, AND THE WIDTH IS A MULTIPLE OF $\sigma(p)$.



SO WE FUDGE A LITTLE AND USE THE ***STANDARD ERROR*** OF $\hat{P}$:

$$SE(\hat{P}) = \frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}$$

IN ITS PLACE.. IT'S CLOSE ENOUGH... IT'S THE BEST WE CAN DO... AND IT CAN EVEN BE THEORETICALLY JUSTIFIED!

NOW THE FORMULA IS

$$.95 = \Pr\left(\hat{p} - 1.96\,SE(\hat{p}) \le p \le \hat{p} + 1.96\,SE(\hat{p})\right)$$

AGAIN, THIS EQUATION DESCRIBES THE PROBABILITY THAT THE TRUE, FIXED POPULATION PROPORTION FALLS WITHIN THE *RANDOM INTERVAL*

$$(\hat{p} - 1.96\,SE(\hat{p}),\ \hat{p} + 1.96\,SE(\hat{p})).$$

IF WE SAMPLED REPEATEDLY, THESE INTERVALS WOULD *COVER* p 95% OF THE TIME

NOW OUR PROBABILITY CALCULATION IS DONE, AND IT'S TIME FOR...

Step Two:

THE DETECTIVE WORK IN A REAL POLL. HOLMES TAKES JUST ONE SIMPLE RANDOM SAMPLE OF 1000 VOTES, FINDS $\hat{p} = .550$, AND WANTS TO INFER p.

HE MAKES USE OF STEP ONE TO COMPUTE

$$SE(\hat{p}) = \sqrt{\frac{(.55)(.45)}{1000}} = .0157$$

HE CONCLUDES THAT WE CAN HAVE 95% CONFIDENCE THAT p IS WITHIN THE RANGE

$$\hat{p} \pm 1.96\,SE(\hat{p})$$

$$= .550 \pm (1.96)(.0157)$$

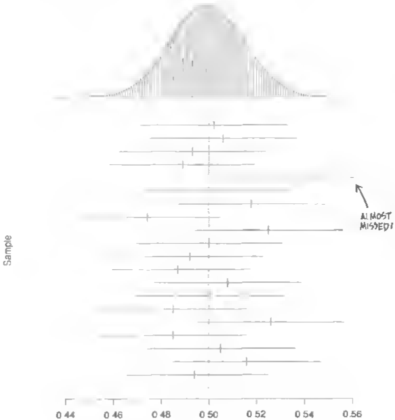
$$= .550 \pm .031$$

THIS IS WHAT POLLS MEAN WHEN THEY REFER TO THEIR "MARGIN OF ERROR." IN THIS CASE, HOLMES FOUND THAT

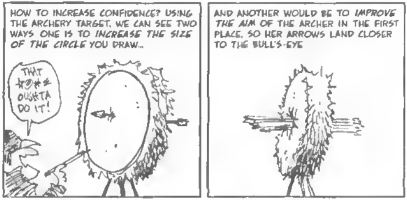
$$.519 \le p \le .581,$$

IN OTHER WORDS THAT $p = 55\%$ WITH A 3% MARGIN OF ERROR. (POLLS TYPICALLY USE A 95% CONFIDENCE LEVEL.)

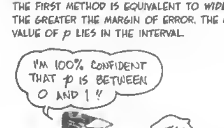
THIS PAGE SHOWS THE RESULTS OF A COMPUTER SIMULATION OF TWENTY SAMPLES OF SIZE $n = 1000$. WE ASSUMED THAT THE TRUE VALUE OF $p = .5$. AT THE TOP YOU SEE THE SAMPLING DISTRIBUTION OF $\hat{p}$ (NORMAL, WITH MEAN p AND $\sigma = \sqrt{\frac{p(1-p)}{n}}$). BELOW ARE THE 95% CONFIDENCE INTERVALS FROM EACH SAMPLE. ON AVERAGE, ONE OUT OF TWENTY (OR 5%) OF THESE INTERVALS WILL NOT COVER THE POINT $p = .5$.



ALTHOUGH 95% CONFIDENCE IS GOOD ENOUGH FOR NEWSPAPER POLLS, IT ISN'T GOOD ENOUGH FOR *SENATOR ASTUTE* HE WANTS 99%!



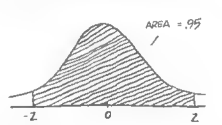
THE FIRST METHOD IS EQUIVALENT TO *WIDENING THE CONFIDENCE INTERVAL*. THE GREATER THE MARGIN OF ERROR, THE MORE CERTAIN YOU ARE THE TRUE VALUE OF p LIES IN THE INTERVAL.



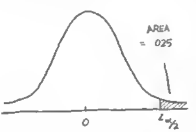
MAYBE IT'S TIME TO SEE EXACTLY HOW WE FIND THE ENDS OF THESE CONFIDENCE INTERVALS

THE RELEVANT NUMBER HERE WE USUALLY CALL α. IT MEASURES THE DIFFERENCE BETWEEN THE DESIRED CONFIDENCE LEVEL AND CERTAINTY. FOR EXAMPLE, WHEN THE CONFIDENCE LEVEL IS 95%, OR 0.95, α IS .05 SO WE SPEAK OF THE $(1-\alpha)\cdot 100\%$ CONFIDENCE LEVEL.

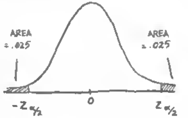
FINDING THE $(1-\alpha)\cdot 100\%$ CONFIDENCE INTERVAL MEANS LOOK AT A STANDARD NORMAL CURVE, AND FIND THE POINTS $\pm z$ BETWEEN WHICH THE AREA IS $1-\alpha$.



THIS POINT, CALLED $z_{\frac{\alpha}{2}}$, IS THE Z-VALUE BEYOND WHICH THE AREA IS $.025 = \frac{\alpha}{2}$



THAT'S BECAUSE WE'RE CHOPPING OFF "TAILS" AT BOTH ENDS OF THE CURVE, WHICH HAVE A TOTAL AREA OF $\frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$



WE CAN FIND $z_{\alpha/2}$ STRAIGHT FROM THE STANDARD NORMAL TABLE (PAGE 84). IT'S THE POINT WITH THE PROPERTY

$$Pr(z \geq z_{\alpha/2}) = \frac{\alpha}{2}$$

IN PARTICULAR,

$$Pr(z \geq z_{.025}) = .025$$

z	2.5	-2.4	-2.3	2.2	-2.1
F(z)	0.006	0.008	0.011	0.014	0.018
z	-2.0	-1.9	1.8	-1.7	1.6
F(z)	0.023	0.029	0.036	0.045	0.055
z	-1.5				
F(z)	0.067	[illegible]			

HERE'S A LITTLE TABLE OF THE CRITICAL VALUES FOR VARIOUS LEVELS OF CONFIDENCE..

$1-\alpha$	.80	.90	.95	.99
α	.20	.10	.05	.01
$\alpha/2$	.10	.05	.025	.005
$z_{\frac{\alpha}{2}}$	1.28	1.64	1.96	2.58

TO MAKE A 99% CONFIDENCE INTERVAL, WE USE THAT TABLE TO WRITE

$$.99 = Pr(\hat{p} - 2.58SE(\hat{p}) \leq p \leq \hat{p} + 2.58SE(\hat{p}))$$

WHICH WE SLOPPILY ABBREVIATE AS

$$p = \hat{p} \pm 2.58\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= .55 \pm 2.58\sqrt{\frac{(.55)(.45)}{1000}}$$

$$= .55 \pm .041$$

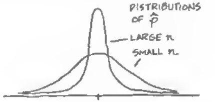
WITH 99% CONFIDENCE.

WIDENING THE INTERVAL IS ONE WAY TO INCREASE OUR CONFIDENCE IN THE RESULT. AS WE MENTIONED, ANOTHER WAY WOULD BE TO SHOOT OUR ARROWS ***MORE ACCURATELY.*** IF WE KNEW THAT THE ARCHER GOT 95% OF HER ARROWS WITHIN ***1 CM*** OF THE BULL'S-EYE, OUR ESTIMATES COULD BE A LOT SHARPER!

HOW DO WE DO THIS? BY INCREASING THE SAMPLE SIZE! THE WIDTH OF THE CONFIDENCE INTERVAL DEPENDS ON THE SAMPLE SIZE: THE INTERVAL HAS THE FORM $\hat{p} \pm E$, WHERE E, THE ERROR, IS GIVEN BY

$$E = z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

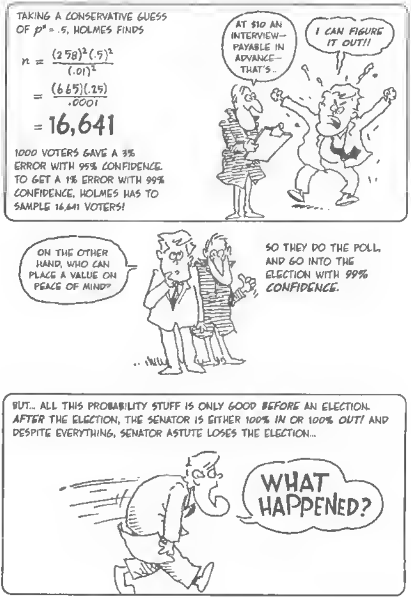
SO THE BIGGER WE MAKE n, THE SMALLER THE ERROR. (E.G., QUADRUPLING n HALVES THE INTERVAL WIDTH)



ASTUTE ASKS HOLMES TO GIVE HIM A SMALL ERROR WITH HIGH CONFIDENCE—SAY 99% CONFIDENCE WITH E = ±.01. HOLMES SOLVES FOR n.

$$n = \frac{z_{\frac{\alpha}{2}}^{2}\, p^{*}(1-p^{*})}{E^{2}}$$

(WHERE p^{*} IS A GUESS AT THE TRUE PROPORTION p—REMEMBER, WE HAVEN'T TAKEN THE SAMPLE YET!)



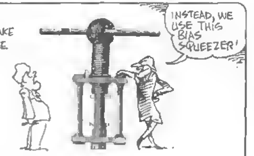
TAKING A CONSERVATIVE GUESS OF $p^* = .5$, HOLMES FINDS
$n = \frac{(2.58)^2(.5)^2}{(.01)^2}$
$= \frac{(6.65)(.25)}{.0001}$
$= 16,641$
1000 VOTERS GAVE A 3% ERROR WITH 95% CONFIDENCE. TO GET A 1% ERROR WITH 99% CONFIDENCE, HOLMES HAS TO SAMPLE 16,641 VOTERS!
AT $10 AN INTERVIEW—PAYABLE IN ADVANCE—THAT'S..
I CAN FIGURE IT OUT!!
ON THE OTHER HAND, WHO CAN PLACE A VALUE ON PEACE OF MIND?
SO THEY DO THE POLL, AND GO INTO THE ELECTION WITH 99% CONFIDENCE.
BUT... ALL THIS PROBABILITY STUFF IS ONLY GOOD BEFORE AN ELECTION. AFTER THE ELECTION, THE SENATOR IS EITHER 100% IN OR 100% OUT! AND DESPITE EVERYTHING, SENATOR ASTUTE LOSES THE ELECTION...
WHAT HAPPENED?

WHAT HAPPENED IS THAT POLITICIANS ARE NOT ELECTED BY POLLS!

SOME PROBLEMS WITH POLLS, AS OPPOSED TO ELECTIONS:

THERE IS NO WAY FOR A POLLSTER TO GET INSIDE A POTENTIAL VOTER'S HEAD AND KNOW IF SHE'S GOING TO VOTE, IF SHE'S LYING, OR IF SHE'S GOING TO CHANGE HER MIND BEFORE ELECTION DAY. LARGE SAMPLE SIZES CANNOT REDUCE THESE KINDS OF ERRORS.

SINCE THESE ERRORS CAN BE LARGE, IT SELDOM PAYS TO TAKE A VERY LARGE RANDOM SAMPLE.



IN THE LAST FIVE PRESIDENTIAL ELECTIONS, THE GALLUP POLL HAS INTERVIEWED FEWER THAN 4,000 VOTERS FOR EACH ELECTION. YET IN ALL FIVE ELECTIONS, THE GALLUP ORGANIZATION'S ERRORS IN PREDICTING THE PRESIDENTIAL ELECTION OUTCOME HAVE BEEN LESS THAN 2%.

THEIR SUCCESS IS DUE TO THEIR USE OF ESTIMATORS THAT ACCOUNT FOR NON-RESPONSE, AND THEY SCREEN OUT ELIGIBLE VOTERS WHO ARE NOT LIKELY TO VOTE.

TO SUMMARIZE, ESTIMATED PROPORTION = TRUE PROPORTION + BIAS + RANDOM SAMPLING ERROR. EVEN POLLSTERS HAVE LIMITED FUNDS. THEY WISELY CHOOSE TO SPEND THEIR MONEY *REDUCING* BIAS, RATHER THAN INCREASING THE SAMPLES BEYOND 4,000 VOTERS.

Confidence Intervals for μ

UP TO NOW, WE'VE BEEN LOOKING AT CONFIDENCE INTERVALS FOR A PROPORTION p OF A POPULATION. EXACTLY THE SAME REASONING WORKS FOR THE POPULATION MEAN μ.

IN THE LAST CHAPTER (P 105), WE SAW THAT THE DISTRIBUTION OF SAMPLE MEANS $\bar{X}$ IS APPROXIMATELY *NORMAL*, CENTERED ON THE ACTUAL POPULATION MEAN μ, WITH STANDARD DEVIATION $\sigma/\sqrt{n}$, WHERE σ IS THE POPULATION STANDARD DEVIATION. SO, FOR LARGE n,

$$.95 = Pr(-1.96 \leq Z \leq 1.96)$$

$$\approx Pr\left(-1.96 \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq 1.96\right)$$

AGAIN, NOT KNOWING σ, WE REPLACE σ WITH s, THE SAMPLE STANDARD DEVIATION:

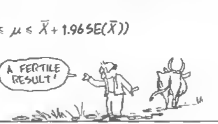
$$.95 \approx Pr\left(-1.96 \leq \frac{\bar{X}-\mu}{s/\sqrt{n}} \leq 1.96\right)$$

THE TERM $s/\sqrt{n}$ IS CALLED THE ***SAMPLE STANDARD ERROR***, AND WRITTEN $SE(\bar{X})$. WE CONCLUDE THAT

$$.95 \approx Pr(\bar{X} - 1.96SE(\bar{X}) \leq \mu \leq \bar{X} + 1.96SE(\bar{X}))$$

WHERE

$$SE(\bar{X}) = \frac{s}{\sqrt{n}}$$



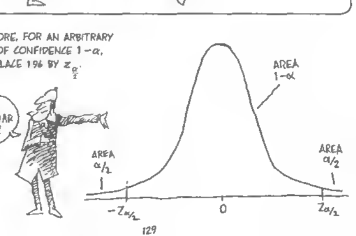
JUST AS BEFORE, WE HAVE FOUND THAT THE RANDOM INTERVAL

$$\bar{X} \pm 1.96\,SE(\bar{X})$$

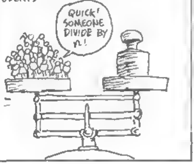
COVERS THE TRUE MEAN, μ, WITH PROBABILITY .95... SO NOW WE CAN CALL IN SHERLOCK HOLMES TO MAKE A STATISTICAL INFERENCE BASED ON A SINGLE SAMPLE OF SIZE n WITH MEAN $\bar{x}$.

HE (AND WE) ARE 95% CONFIDENT THAT THE MEAN μ IS WITHIN THE INTERVAL $\bar{x} \pm 1.96\,SE(\bar{X})$.

AS BEFORE, FOR AN ARBITRARY LEVEL OF CONFIDENCE $1-\alpha$, WE REPLACE 1.96 BY $z_{\frac{\alpha}{2}}$.



LET'S REVISIT THE STUDENT WEIGHT DATA FROM CHAPTER 2, ASSUMING THAT THE $n = 92$ STUDENTS WERE A SIMPLE RANDOM SAMPLE OF ALL PENN STATE STUDENTS



THE SAMPLE MEAN $\bar{x}$ WAS 145.2 LBS AND SAMPLE STANDARD DEVIATION s WAS 23.7. SO THE STANDARD ERROR IS

$$SE(\bar{x}) = \frac{23.7}{\sqrt{92}} = \mathbf{2.47}$$

AND WE NOW HAVE 95% CONFIDENCE THAT THE MEAN WEIGHT OF ALL PENN STATE STUDENTS FALLS IN THE INTERVAL

$$\bar{x} \pm 1.96 SE(\bar{x})$$
$$= 145.2 \pm (1.96)(2.47)$$
$$= \mathbf{145.2 \pm 4.8} \text{ POUNDS}$$

TO SUMMARIZE: FOR A SIMPLE RANDOM SAMPLE (SRS) OF LARGE SIZE, THE $(1-\alpha)\cdot 100\%$ CONFIDENCE INTERVAL IS:

POPULATION MEAN μ

$$\mu = \bar{x} \pm z_{\frac{\alpha}{2}} SE(\bar{x})$$

WHERE $SE(\bar{x}) = s/\sqrt{n}$

POPULATION PROPORTION p

$$p = \hat{p} \pm z_{\frac{\alpha}{2}} SE(\hat{p})$$

WHERE $SE(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

THE SIZE OF BOTH INTERVALS IS CONTROLLED BY THE LEVEL OF CONFIDENCE $(1-\alpha)\cdot 100\%$ AND THE SAMPLE SIZE, n.

Student's t (again!)

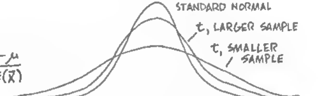
AS WE SAW IN CHAPTER 6, THE STATISTIC

$$\frac{\bar{X}-\mu}{SE(\bar{X})}$$

HAS AN APPROXIMATELY NORMAL DISTRIBUTION ONLY WHEN IT IS COMPUTED USING A *LARGE SAMPLE*. FOR SMALL SAMPLES (n=5, 10, 25...), THIS IS NO LONGER THE CASE, AND WE HAVE TO USE THE STUDENT'S t.

LET'S LOOK AT t A LITTLE MORE CLOSELY. WE MENTIONED THAT THE t DISTRIBUTION IS MORE SPREAD OUT THAN THE NORMAL, AND THAT THE AMOUNT OF SPREAD DEPENDS ON THE SAMPLE SIZE.

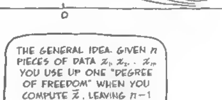
$$t = \frac{\bar{X}-\mu}{SE(\bar{X})}$$



WHAT ITS DISCOVERER GOSSET DID WAS TO *QUANTIFY* THIS RELATIONSHIP. IF n IS THE SAMPLE SIZE, HE SAID, THEN CALL $n-1$ THE NUMBER OF

degrees of freedom

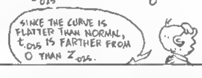
OF THE SAMPLE.



GOSSET COMPUTED TABLES OF THE t DISTRIBUTION FOR DIFFERENT SAMPLE SIZES—I.E., DEGREES OF FREEDOM. WE REPEAT, THE *MORE DEGREES OF FREEDOM*, THE CLOSER t BECOMES TO THE *STANDARD NORMAL*.

KNOWING THE SAMPLE SIZE n, WE CHOOSE THE t DISTRIBUTION WITH $n-1$ DEGREES OF FREEDOM.

AS WITH THE z DISTRIBUTION (I.E., THE STANDARD NORMAL), WE GET A 95% CONFIDENCE LEVEL BY FINDING THE CRITICAL VALUE $t_{.025}$ BEYOND WHICH THE AREA UNDER THE CURVE IS .025.



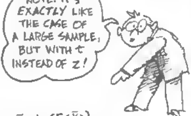
FOR A $(1-\alpha)\cdot 100\%$ CONFIDENCE INTERVAL, WE FIND THE CRITICAL VALUE $t_{\frac{\alpha}{2}}$ SUCH THAT $Pr(t \geq t_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$. HERE IS A SHORT TABLE OF CRITICAL VALUES FOR THE t DISTRIBUTION:

	$1-\alpha$	.80	.90	.95	.99
	α	.20	.10	.05	.01
	$\alpha/2$	.10	.05	.025	.005
DEGREES OF FREEDOM	1	3.09	6.31	12.71	63.66
	10	1.37	1.81	2.23	4.14
	30	1.31	1.70	2.04	2.75
	100	1.29	1.66	1.98	2.63
	∞	1.28	1.65	1.96	2.58

EACH COLUMN REPRESENTS A FIXED LEVEL OF CONFIDENCE, WITH INCREASING NUMBERS OF DEGREES OF FREEDOM. THE HIGHER THE DEGREES OF FREEDOM, THE CLOSER THE CRITICAL VALUE GETS TO $z_{\alpha/2}$, THE CRITICAL VALUE OF THE NORMAL DISTRIBUTION

WE DERIVE THE WIDTH OF OUR CONFIDENCE INTERVAL DIRECTLY FROM THE DEFINITION OF t:

$$t = \frac{\bar{X}-\mu}{SE(\bar{X})}$$



THEN, FOR CONFIDENCE LEVEL $(1-\alpha)\cdot 100\%$,

$$(1-\alpha) = Pr\left(\bar{x} - t_{\frac{\alpha}{2}} SE(\bar{X}) \leq \mu \leq \bar{x} + t_{\frac{\alpha}{2}} SE(\bar{X})\right)$$

FROM WHICH WE INFER: GIVEN A SINGLE SAMPLE OF SIZE n AND MEAN $\bar{x}$, WE CAN BE $(1-\alpha)\cdot 100\%$ CONFIDENT THAT THE POPULATION MEAN μ FALLS IN THE RANGE

$$\mu = \bar{x} \pm t_{\frac{\alpha}{2}} SE(\bar{x})$$

WHERE $SE(\bar{x}) = s/\sqrt{n}$ AND $t_{\frac{\alpha}{2}}$ IS THE CRITICAL VALUE OF THE t DISTRIBUTION WITH $n-1$ DEGREES OF FREEDOM.

NOTE: STRICTLY SPEAKING, THE DERIVATION OF THE t DISTRIBUTION DEPENDED ON THE ASSUMPTION THAT THE SAMPLE WAS FROM A NORMAL POPULATION. IN PRACTICE, CONFIDENCE INTERVALS BASED ON THE t WORK REASONABLY WELL, EVEN WHEN THE POPULATION DISTRIBUTION IS ONLY APPROXIMATELY MOUND-SHAPED.

example: SUPPOSE *CHAMELEON MOTORS* HAS TO CRASH TEST ITS CARS TO DETERMINE THE AVERAGE REPAIR COST OF A 10 M.P.H. HEAD-ON COLLISION. THIS IS EXPENSIVE! THEY DECIDE TO TRY IT ON JUST *FIVE* CHAMELEONS.

THEY FIND THE DAMAGE DATA TO BE \$150, \$400, \$720, \$500, AND \$930.

THE SAMPLE MEAN.

$\bar{x} = \$540$

THE STANDARD DEVIATION

$s = \$299$

YOU CAN CHECK s WITH A HAND CALCULATOR. IT'S

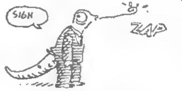
$$\sqrt{\frac{1}{4}\left((150-540)^2+(400-540)^2+(720-540)^2+(500-540)^2+(930-540)^2\right)}$$

SO WHERE CAN WE PLACE THE MEAN WITH 95% CONFIDENCE? WE FIND OUR CRITICAL VALUE $t_{.025}$ WITH 4 DEGREES OF FREEDOM

	$1-\alpha$	.80	.90	.95	.99
	α	.20	.10	.05	.01
	$\alpha/2$	.10	.05	.025	.005
DEGREES OF FREEDOM	1	3.09	6.31	12.71	63.66
	2	1.89	2.92	4.30	9.92
	3	1.64	2.35	3.18	5.84
	4	1.53	2.13	2.78	4.60
	5	1.48	2.01	2.57	4.03

AND PLUG IT IN:

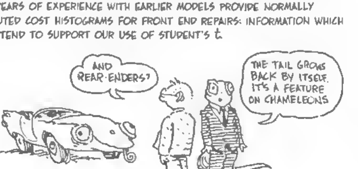
$$\mu = \bar{x} \pm 2.78\, s/\sqrt{n}$$
$$= 540 \pm 2.78\left(299/\sqrt{5}\right)$$
$$= \mathbf{540 \pm 372}$$



SO THE BEST WE CAN SAY WITH 95% CONFIDENCE IS THAT THE AVERAGE DAMAGE WILL LIE BETWEEN \$168 AND \$912.

THE COMPANY CAN EITHER BE SATISFIED WITH THAT, OR DO FURTHER TESTS...

TO COMPUTE THIS CONFIDENCE INTERVAL USING STUDENT'S t, WE HAVE MADE AN ***UNSTATED ASSUMPTION:*** WE ASSUMED THAT CRASH REPAIR COSTS ARE APPROXIMATELY ***NORMALLY DISTRIBUTED,*** I.E., IF WE CRASHED 1000 CHAMELEONS, THE HISTOGRAM OF REPAIR COSTS WOULD BE SYMMETRICAL AND MOUND-SHAPED. WE ***CAN NOT KNOW THIS*** FROM 5 DATA POINTS ALONE... BUT MAYBE YEARS OF EXPERIENCE WITH EARLIER MODELS PROVIDE NORMALLY DISTRIBUTED COST HISTOGRAMS FOR FRONT END REPAIRS: INFORMATION WHICH WOULD TEND TO SUPPORT OUR USE OF STUDENT'S t.



TO SUM UP (1), WE NOW HAVE THREE SIMPLE RECIPES FOR FINDING CONFIDENCE INTERVALS. FOR PROPORTIONS, OR MEANS WITH LARGE SAMPLE SIZES, WE LOOK UP $z_{\frac{\alpha}{2}}$ IN A NORMAL TABLE. FOR MEANS OF SMALL SAMPLE SIZES (SAY $n \leq 30$), WE FIND $t_{\frac{\alpha}{2}}$ IN THE t TABLE.

IN ALL CASES, THE WIDTH OF THE INTERVAL IS THAT CRITICAL VALUE TIMES THE STANDARD ERROR:

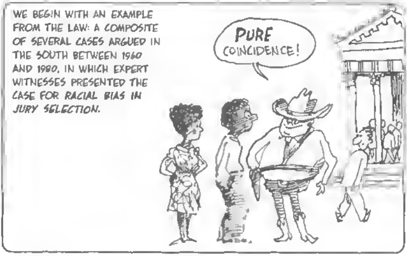
$$z_{\frac{\alpha}{2}} SE(\hat{p}) \qquad z_{\frac{\alpha}{2}} SE(\overline{X}) \qquad t_{\frac{\alpha}{2}} SE(\overline{X})$$

AND EACH OF THOSE STANDARD ERRORS IS PROPORTIONAL TO THAT MAGIC NUMBER:

$$\frac{1}{\sqrt{n}}.$$

♦Chapter 8♦
HYPOTHESIS TESTING

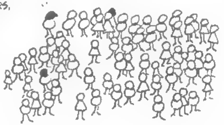
NOW WE ENTER A NEW AREA... GOVERNMENT, BUSINESS, AND THE HARD AND SOFT SCIENCES ALL USE AND OFTEN ABUSE THESE TESTS OF SIGNIFICANCE. IT'S ALL ABOUT ANSWERING THE QUESTION, ***"COULD THESE OBSERVATIONS REALLY HAVE OCCURRED BY CHANCE?"***



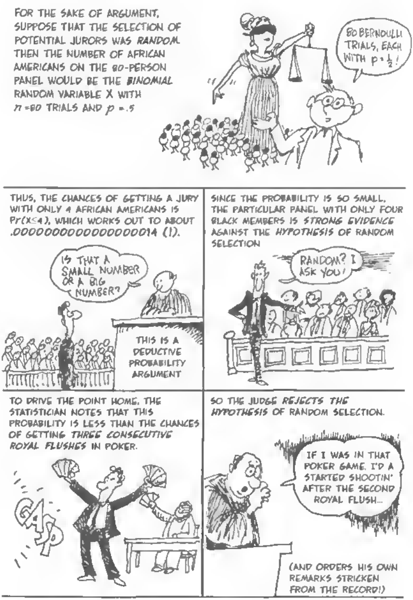
PANELS OF JURORS ARE THEORETICALLY DRAWN AT RANDOM FROM A LIST OF ELIGIBLE CITIZENS. HOWEVER, IN SOUTHERN STATES IN THE '50S AND '60S, FEW AFRICAN AMERICANS WERE FOUND ON JURY PANELS, SO SOME DEFENDANTS CHALLENGED THE VERDICTS. ON APPEAL, AN EXPERT STATISTICAL WITNESS GAVE THIS EVIDENCE:

1) 50% OF ELIGIBLE CITIZENS WERE AFRICAN AMERICAN.

2) ON AN 80-PERSON PANEL OF POTENTIAL JURORS, ONLY *FOUR* WERE AFRICAN AMERICANS.



COULD THIS BE THE RESULT OF *PURE CHANCE?*

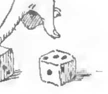

FOR THE SAKE OF ARGUMENT, SUPPOSE THAT THE SELECTION OF POTENTIAL JURORS WAS RANDOM. THEN THE NUMBER OF AFRICAN AMERICANS ON THE 80-PERSON PANEL WOULD BE THE BINOMIAL RANDOM VARIABLE X WITH n = 80 TRIALS AND p = .5
80 BERNOULLI TRIALS, EACH WITH p = 1/2!
THUS, THE CHANCES OF GETTING A JURY WITH ONLY 4 AFRICAN AMERICANS IS Pr(X≤4), WHICH WORKS OUT TO ABOUT .0000000000000000014 (!).
IS THAT A SMALL NUMBER OR A BIG NUMBER?
THIS IS A DEDUCTIVE PROBABILITY ARGUMENT
SINCE THE PROBABILITY IS SO SMALL, THE PARTICULAR PANEL WITH ONLY FOUR BLACK MEMBERS IS STRONG EVIDENCE AGAINST THE HYPOTHESIS OF RANDOM SELECTION
RANDOM? I ASK YOU!
TO DRIVE THE POINT HOME, THE STATISTICIAN NOTES THAT THIS PROBABILITY IS LESS THAN THE CHANCES OF GETTING THREE CONSECUTIVE ROYAL FLUSHES IN POKER.
GASP
SO THE JUDGE REJECTS THE HYPOTHESIS OF RANDOM SELECTION.
IF I WAS IN THAT POKER GAME, I'D A STARTED SHOOTIN' AFTER THE SECOND ROYAL FLUSH...
(AND ORDERS HIS OWN REMARKS STRICKEN FROM THE RECORD!)

LET'S FOLLOW THE PROCESS AGAIN TO SORT OUT THE *FOUR FORMAL STEPS* OF STATISTICAL HYPOTHESIS TESTING

Step 1. *FORMULATE ALL HYPOTHESES.*

H_o, THE *NULL HYPOTHESIS,* IS USUALLY THAT THE OBSERVATIONS ARE THE RESULT PURELY OF *CHANCE*

H_a, THE *ALTERNATE HYPOTHESIS,* IS THAT THERE IS A REAL EFFECT, THAT THE OBSERVATIONS ARE THE RESULT OF THIS REAL EFFECT, PLUS CHANCE VARIATION.

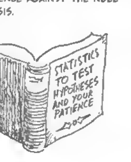


IN THE COURT CASE, H_o SAYS THE JURY WAS *RANDOMLY CHOSEN* FROM THE WHOLE POPULATION. AFRICAN AMERICANS HAVE PROBABILITY $p = .50$ OF BEING CHOSEN.

H_a SAYS THAT AFRICAN AMERICANS ARE LESS LIKELY THAN THEIR PROPORTION IN THE POPULATION TO BE SELECTED FOR A JURY PANEL. $p < .50$.

Step 2. THE *TEST STATISTIC.*

IDENTIFY A STATISTIC THAT WILL ASSESS THE EVIDENCE AGAINST THE NULL HYPOTHESIS.



IN THE COURT CASE, THE TEST STATISTIC IS THE BINOMIAL RANDOM VARIABLE X WITH $p = .50$ AND $n = 80$

Step 3. *P-VALUE:*

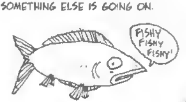
A PROBABILITY STATEMENT WHICH ANSWERS THE QUESTION: IF THE NULL HYPOTHESIS WERE TRUE, THEN WHAT IS THE PROBABILITY OF OBSERVING A TEST STATISTIC AT LEAST AS EXTREME AS THE ONE WE OBSERVED?

Step 4. COMPARE THE P-VALUE TO A FIXED *SIGNIFICANCE LEVEL*, α

α ACTS AS A CUT-OFF POINT BELOW WHICH WE AGREE THAT AN EFFECT IS STATISTICALLY SIGNIFICANT. THAT IS, IF

$$\text{P-VALUE} \leq \alpha$$

THEN WE *RULE OUT THE NULL HYPOTHESIS* H_0 AND AGREE THAT SOMETHING ELSE IS GOING ON.



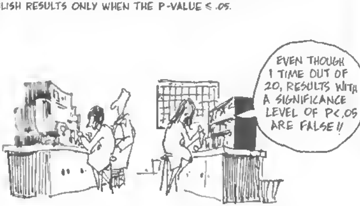
IN THE EXAMPLE, THE P-VALUE WAS

$$Pr(X \leq 4 \mid p = .50 \text{ AND } n = 80)$$
$$= 1.4 \times 10^{-18}$$

WE COMPUTED THIS P-VALUE THE MODERN WAY, USING A STATISTICAL SOFTWARE PACKAGE

IN THE JURY CASE, THE STATISTICIAN TOOK α TO BE 3.6×10^{-18}, THE CHANCES OF BEING DEALT THREE ROYAL FLUSHES IN A ROW.

IN SCIENTIFIC WORK, A FIXED α-LEVEL OF .05 OR .01 IS OFTEN USED. THESE FIXED LEVELS ARE A HOLDOVER ARTIFACT FROM THE PRE-COMPUTER ERA, WHEN WE HAD TO REFER TO TABLES, WHICH WERE PRINTED ONLY FOR SELECTED CRITICAL VALUES. STILL, MANY SCIENTIFIC JOURNALS CONTINUE TO PUBLISH RESULTS ONLY WHEN THE P-VALUE $\leq$ .05.



IN LEGAL PROCEEDINGS, THE STANDARD IS MORE FLEXIBLE...

LARGE SAMPLE SIGNIFICANCE TEST FOR PROPORTIONS

THE JURY EXAMPLE WAS A SPECIAL CASE OF A GENERAL PROBLEM. THE NULL HYPOTHESIS HAD THE FORM $p = p_0$, WHERE p_0 WAS SOME PROBABILITY (IN THIS CASE, .5). NOW LET'S LOOK AT SUCH PROBLEMS GENERALLY. LET'S *TEST THE HYPOTHESIS* $p = p_0$.

AS USUAL, WE IMAGINE WE HAVE A BIG POPULATION... WE OBSERVE A LARGE SAMPLE... AND WE FIND THAT SOME CHARACTERISTIC OCCURS WITH PROBABILITY $\hat{p}$.

BASED ON THIS OBSERVATION, WE WANT TO KNOW IF THE TRUE POPULATION PROBABILITY IS (FOR INSTANCE) LARGER THAN SOME OTHER VALUE p_0. FOR EXAMPLE, SENATOR ASTUTE, HAVING FOUND A $\hat{p}$ OF 55, WOULD LIKE TO KNOW THAT $p > 5$, A WINNING MAJORITY.

Step 1.

THE *NULL* HYPOTHESIS IS

$$H_0 : p = p_0$$

THE *ALTERNATE* HYPOTHESIS DEPENDS ON THE DIRECTION OF THE EFFECT WE ARE LOOKING FOR. IN SENATOR ASTUTE'S CASE,

$$H_a : p > p_0$$

BUT IN OTHER CASES, THE ALTERNATE HYPOTHESIS MIGHT WELL BE

$$H_a : p < p_0$$

OR

$$H_a : p \neq p_0$$

FOR EXAMPLE, IN THE JURY SELECTION EXAMPLE, THE ALTERNATIVE HYPOTHESIS WAS

$$H_a : p < 0.5$$

AND AT OTHER TIMES, WE ARE INTERESTED IN KNOWING THAT p IS DIFFERENT FROM SOME VALUE p_0. FOR INSTANCE, IN TESTING FOR A FAIR COIN, WE HAVE AN ALTERNATE HYPOTHESIS OF

$$H_a : p \neq 0.5$$

BUT HAVE NO *A PRIORI* OPINION ABOUT WHETHER HEADS OR TAILS WILL COME UP MORE OFTEN.

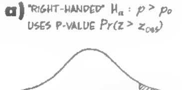
Step 2. THE TEST STATISTIC IS

$$z_{OBS} = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)}/\sqrt{n}}$$

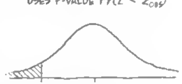
WHICH MEASURES HOW FAR p DEVIATES FROM p_0. UNDER THE NULL HYPOTHESIS, z_{OBS} HAS THE STANDARD NORMAL DISTRIBUTION.

Step 3. THE P-VALUE DEPENDS ON WHICH ALTERNATE HYPOTHESIS IS RELEVANT.

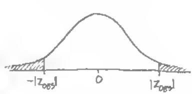
a) "RIGHT-HANDED" $H_a : p > p_0$ USES P-VALUE $Pr(z > z_{OBS})$



b) "LEFT-HANDED" $H_a : p < p_0$ USES P-VALUE $Pr(z < z_{OBS})$



c) "TWO-SIDED" $H_a : p \neq p_0$ USES P-VALUE $Pr(|z| > |z_{OBS}|)$



IN THE CASE OF SENATOR ASTUTE:

1) THE HYPOTHESES ARE

$H_0 : p = .5$

$H_a : p > .5$

2) HIS TEST STATISTIC IS

$$z_{OBS} = \frac{.55-.50}{\sqrt{(.5)(.5)}/\sqrt{1000}} = 3.16$$

3) HIS P-VALUE IS

$$Pr(z > z_{OBS}) = Pr(z \geq 3.16) = .0008$$

(FROM THE NORMAL TABLE).

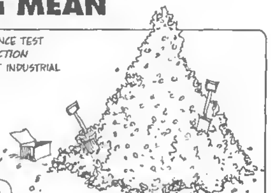
4) ASTUTE, BEING FAIRLY CONSERVATIVE, TAKES A SIGNIFICANCE LEVEL α OF .01 AND OBSERVES THAT

$$Pr(z > z_{OBS}) = .0008 < \alpha$$

THE SENATOR THUS REJECTS THE NULL HYPOTHESIS, AND HE (AND HIS BACKERS) NOW FEEL CERTAIN HE'S IN THE LEAD

LARGE SAMPLE TEST FOR THE POPULATION MEAN

HERE IS HOW A SIGNIFICANCE TEST MIGHT BE USED IN *INSPECTION SAMPLING*, AN IMPORTANT INDUSTRIAL APPLICATION.



NEW AGE GRANOLA INC. CLAIMS THAT THE AVERAGE WEIGHT OF ITS CEREAL BOXES IS AT LEAST *16 OZ.* THE *GENUINE GROCERY CORPORATION* WILL SEND BACK A SHIPMENT IF THE AVERAGE WEIGHT IS ANY LESS

BUT OF COURSE GENUINE GROCERY HAS *NO INTENTION* OF WEIGHING EVERY BOX IN A SHIPMENT. THEY'RE GOING TO USE STATISTICS!

FIRST, THEY CHOOSE THEIR HYPOTHESES.

$$H_0: \mu = 16 \text{ OZ.}$$

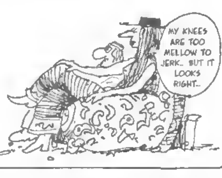
$$H_a: \mu < 16 \text{ OZ.}$$

REJECTING THE NULL HYPOTHESIS MEANS REFUSING THE GRANOLA

NEXT, THEY CHOOSE A TEST STATISTIC. BY NOW, IT SHOULD BE PRETTY MUCH A KNEE-JERK REACTION TO KNOW THAT THE SAMPLE SPREAD FROM THE MEAN IS

$$\frac{\bar{X}-\mu_0}{SE(\bar{X})} = \frac{\bar{X}-\mu_0}{s/\sqrt{n}}$$

WHERE s IS THE SAMPLE STANDARD DEVIATION UNDER THE NULL HYPOTHESIS. THIS APPROXIMATES THE STANDARD NORMAL WHEN THE SAMPLE IS LARGE, BY THE CENTRAL LIMIT THEOREM.



SKIPPING OVER STEP 3 FOR A MOMENT, THEY SET A SIGNIFICANCE LEVEL. BEING A BUNCH OF DROPPED-OUT SCIENCE MAJORS, THE GROCERS THINK $\alpha = .05$ SOUNDS ABOUT RIGHT.

JUST THEN, A BOXCAR LOADED WITH 10,000 BOXES OF GRANOLA ARRIVES AT THE DOOR

THEY PULL OUT A SIMPLE RANDOM SAMPLE OF 49 BOXES, WEIGH EACH ONE, AND DETERMINE THE SAMPLE'S SUMMARY STATISTICS.

$\bar{x} = 15.90$ oz.

$s = .35$ oz.

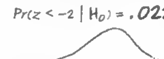
A LITTLE LIGHT--BUT SIGNIFICANTLY SO?

THEY PLUG THE VALUES INTO THE TEST STATISTIC TO FIND

$$Z_{OBS} = \frac{15.9 - 16}{.35/\sqrt{49}} = -2$$

NOW THEY COMPUTE THE P-VALUE

$$Pr(z < -2 \mid H_0) = .0227$$



THIS BEING LESS THAN THE .05 SIGNIFICANCE LEVEL, GENUINE GROCERY *REJECTS* THE NULL HYPOTHESIS, AND THE SHIPMENT.

SMALL SAMPLE
TEST FOR THE POPULATION
MEAN

WE RETURN TO CHAMELEON MOTORS, AND ITS 10 M.P.H. CRASH TEST. THE *RIGHTEOUS INSURANCE COMPANY* WILL INSURE AN AUTO ONLY IF THE MEAN REPAIR COST AFTER A 10 M.P.H. COLLISION IS LESS THAN \$1000. THE COMPANY USES A STANDARD $\alpha = .05$ AS ITS SIGNIFICANCE LEVEL. SO...

H_0: $\mu \geq \$1000$ MEAN COST IS TOO HIGH

H_a: $\mu < \$1000$ MEAN COST IS OK

THE TEST STATISTIC IS THE t DISTRIBUTION

$$t = \frac{\bar{X} - \mu_0}{SE(\bar{X})}$$

WHERE μ_0 IS THE HYPOTHETICAL MEAN OF \$1000

AND WE WANT OUR OBSERVED t VALUE TO LIE TO THE LEFT OF $-t_{.05}$ (BECAUSE LOW $\bar{x}$ IS DESIRABLE, $\bar{x} - \mu_0$ SHOULD BE NEGATIVE, TO SUPPORT H_a).

Degrees of freedom	α = .05	α = .025	α = .005
1	6.31	12.71	63.66
2	2.92	4.30	9.92
3	2.35	3.18	5.84
4	2.13	2.78	4.60
5	2.01	2.57	4.03

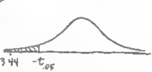
FROM THE TABLE OF CRITICAL t VALUES, WE SEE THAT $t_{.05} = 2.13$, SO WE DECIDE TO REJECT H_0 IF

$$t_{OBS} \le -t_{.05} = -2.13$$

FROM CHAPTER 8, WE HAVE $\bar{x} = \$540$ AND $s = \$299$ FOR A SMALL, FIVE-CAR SAMPLE, SO WE FIND

$$t_{OBS} = \frac{540-1000}{299/\sqrt{5}}$$

$$= -3.44 < -t_{.05}$$

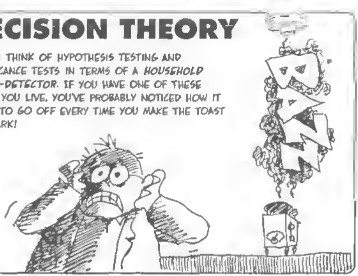


THE CAR PASSES THE TEST... H_0 IS REJECTED... AND THE INSURANCE POLICY IS ISSUED.

THIS IS AN EXAMPLE OF *ACCEPTANCE SAMPLING*. THE NULL HYPOTHESIS IS THAT REPAIR COSTS ARE UNACCEPTABLE, AND THE MOTOR COMPANY IS ASSUMED GUILTY UNTIL IT PRESENTS SUFFICIENT EVIDENCE OF ITS INNOCENCE—I.E., THAT ITS PRODUCT IS WITHIN SPECIFICATIONS

DECISION THEORY

WE CAN THINK OF HYPOTHESIS TESTING AND SIGNIFICANCE TESTS IN TERMS OF A *HOUSEHOLD SMOKE-DETECTOR*. IF YOU HAVE ONE OF THESE WHERE YOU LIVE, YOU'VE PROBABLY NOTICED HOW IT TENDS TO GO OFF EVERY TIME YOU MAKE THE TOAST TOO DARK!



THIS IS WHAT IS CALLED A *TYPE I ERROR*: AN ALARM WITHOUT A FIRE. CONVERSELY, A *TYPE II ERROR* IS A FIRE WITHOUT AN ALARM. EVERY COOK KNOWS HOW TO AVOID A TYPE I ERROR: JUST *REMOVE THE BATTERIES*. UNFORTUNATELY, THIS INCREASES THE INCIDENCE OF TYPE II ERRORS!

SIMILARLY, REDUCING THE CHANCES OF TYPE II ERROR, FOR EXAMPLE BY MAKING THE ALARM HYPERSENSITIVE, CAN INCREASE THE NUMBER OF FALSE ALARMS.

WE CAN SUMMARIZE THIS IN A TWO-BY-TWO *DECISION TABLE.*

	NO FIRE	FIRE
NO ALARM	NO ERROR	TYPE II
ALARM	TYPE I	NO ERROR

NOW THINK OF THE NULL HYPOTHESIS AS THE CONDITION OF *NO FIRE*, WHILE THE ALTERNATE HYPOTHESIS IS THAT A FIRE IS BURNING. THE ALARM CORRESPONDS TO REJECTION OF THE NULL HYPOTHESIS

TRUE STATE

	H_o	H_a
ACCEPT H_o	NO ERROR	TYPE II
REJECT H_o	TYPE I	NO ERROR

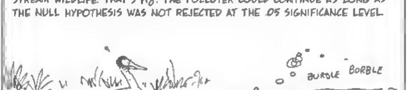
ALL THE SIGNIFICANCE TESTS WE DID EARLIER IN THIS CHAPTER EMPHASIZED THE PROBABILITY OF COMMITTING A TYPE I ERROR—I.E., THE PROBABILITY OF OUR OBSERVATIONS OCCURRING IF H_o WAS TRUE. WE DEMANDED THAT

$$Pr(\text{REJECTING } H_o \mid H_o) = Pr(\text{TYPE I ERROR} \mid H_o) = \alpha$$

$1-\alpha$ MEASURES OUR CONFIDENCE THAT ANY ALARM BELLS WE HEAR ARE GENUINE. HIGH CONFIDENCE MEANS RARELY SETTING OFF FALSE ALARMS

BUT SOMETIMES WHAT WE REALLY WANT TO KNOW IS THE CHANCE OF MAKING A *TYPE II* ERROR! IN OTHER WORDS, HOW SENSITIVE IS OUR "ALARM SYSTEM" WHEN THE ALTERNATE HYPOTHESIS IS *TRUE*?

IN THE PAST, FACTORIES DISCHARGING CHEMICALS INTO WATERWAYS WERE REQUIRED TO SHOW THAT THE DISCHARGE HAD NO EFFECT ON THE DOWNSTREAM WILDLIFE. THAT'S H_0. THE POLLUTER COULD CONTINUE AS LONG AS THE NULL HYPOTHESIS WAS NOT REJECTED AT THE .05 SIGNIFICANCE LEVEL.



SO A POLLUTER, SUSPECTING THAT HE WAS IN VIOLATION OF EPA STANDARDS, WOULD DEVISE AN INEFFECTIVE POLLUTION MONITORING PROGRAM.

THE POLLUTER IS DELIGHTED, SINCE, LIKE OUR SMOKE ALARM WITHOUT A BATTERY, HIS TEST HAS LITTLE OR NO CHANCE OF SETTING OFF AN ALARM.

LET'S FORMALIZE THIS IDEA. TO DESCRIBE THE PROBABILITY OF A *TYPE II ERROR*, WE BREAK OUT ANOTHER GREEK LETTER, BETA, OR β

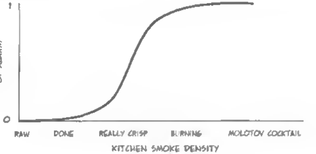
$$\beta = Pr(\text{ACCEPTING } H_0 | H_a)$$
$$= Pr(\text{TYPE II ERROR} | H_a)$$

THE *POWER* OF A TEST IS DEFINED AS $1-\beta$. IT'S

$$Pr(\text{REJECTING } H_0 | H_a).$$

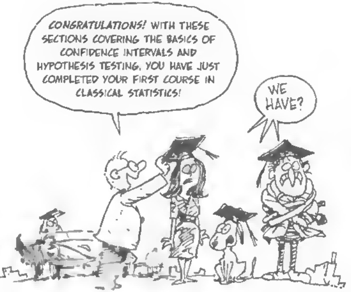
YOU'LL BE HAPPY TO KNOW THE ENVIRONMENTAL REGULATORS HAVE MOVED IN THE DIRECTION OF REQUIRING POLLUTION MONITORING PROGRAMS TO SHOW THAT THEY HAVE A HIGH PROBABILITY OF DETECTING SERIOUS POLLUTION EVENTS. THE REQUIRED *POWER ANALYSIS* OFTEN REVEALS HIDDEN FLAWS IN THE MONITORING PROGRAM.

ONE WAY TO VISUALIZE THE EFFECT OF A TEST'S POWER IS BY GRAPHING THE PROBABILITY OF REJECTING H_0 AGAINST THE ACTUAL STATE OF THE SYSTEM. IN THE CASE OF A SMOKE ALARM, THE PROBABILITY CLIMBS TOWARD 1 AS THE SMOKE GETS THICKER.



FOR THE E.P.A. WATER QUALITY EXAMPLE, THE HORIZONTAL AXIS IS THE TRUE CONCENTRATION OF POLLUTANT IN THE WATER.

HERE ARE THE POWER CURVES FOR THREE MONITORING PROGRAMS: THE *SAVE EVERY LAST GUPPY* (COSTS \$5 MILLION), THE *GOLDEN MEAN* (COSTS \$500,000), AND *DON'T ROCK THE BOAT* (ALSO COSTS \$500,000, BUT THEY PUT ON A GOOD SHOW!). THE HIGHER THE TEST'S POWER, THE STEEPER THE CURVE.

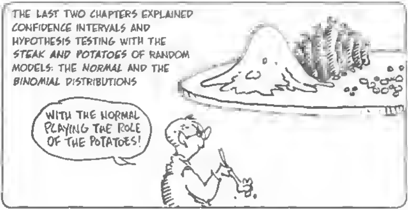


WHY THEN DO YOU HAVE SUCH AN *EMPTY FEELING* IN YOUR STOMACH? BECAUSE, TO USE THESE IDEAS IN ANY PRACTICAL WAY, WE HAVE TO BE ABLE TO APPLY THEM TO A VARIETY OF SITUATIONS WE HAVEN'T EVEN TOUCHED ON YET. THAT IS WHERE WE ARE GOING NEXT, WITH THE *COMPARISON OF TWO POPULATIONS*.

♦Chapter 9♦

COMPARING TWO POPULATIONS

IN WHICH WE LEARN SOME NEW RECIPES USING OLD INGREDIENTS...



BUT WHAT MAKES STATISTICS ALMOST AS CHALLENGING AS COOKING IS THE VARIETY. LIKE AN EXPERT COOK, THE STATISTICIAN CAN "TASTE" THE INGREDIENTS IN A PROBLEM AND THEN FIND THE MOST EFFECTIVE WAY TO COMBINE THEM INTO A STATISTICAL RECIPE.

(THE REASON COOKBOOKS AND STATISTICAL METHODS TEXTS ARE SO HEAVY IS THAT THEY BOTH PROVIDE SOLUTIONS IN A GREAT VARIETY OF SITUATIONS!)

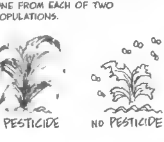
IN THIS CHAPTER, WE'LL USE OUR MEAT AND POTATOES METHODS IN SOME NEW RECIPES THAT WILL HELP US ANSWER THE FOLLOWING QUESTIONS:

DOES TAKING ASPIRIN REGULARLY REDUCE THE RISK OF HEART ATTACK?

DOES A PARTICULAR PESTICIDE INCREASE THE YIELD OF CORN PER ACRE?

DO MEN AND WOMEN IN THE SAME OCCUPATION HAVE DIFFERENT SALARIES?

THE COMMON INGREDIENT IN THESE QUESTIONS IS THIS: THEY CAN BE ANSWERED BY *COMPARING TWO INDEPENDENT RANDOM SAMPLES,* ONE FROM EACH OF TWO POPULATIONS.



AND, AT THE END OF THE CHAPTER, WE'LL LOOK AT A DIFFERENT WAY TO COMPARE TWO MEANS THAT DOESN'T INVOLVE TAKING TWO SIMPLE RANDOM SAMPLES.

Comparing **SUCCESS RATES**
(or failure rates) for two populations.

WE BEGIN WITH AN EXPERIMENT, PART OF A HARVARD STUDY, THAT SOUGHT TO DECIDE THE EFFECTIVENESS OF *ASPIRIN* IN REDUCING HEART ATTACKS. AS IN MOST CLINICAL TRIALS, THE CHANCES THAT ANY ONE INDIVIDUAL GETS THE DISEASE—IN THIS CASE, A HEART ATTACK—IS VERY SMALL IN ANY GIVEN YEAR BUT WE WANT ANSWERS QUICKLY! WHAT DO WE DO?

THE SIMPLE, BUT EXPENSIVE, SOLUTION IS TO TEST A LARGE NUMBER OF INDIVIDUALS IN A SHORT TIME. IN THIS STUDY, 22,071 SUBJECTS (ALL VOLUNTEER DOCTORS) WERE RANDOMLY ASSIGNED TO *TWO GROUPS*.

GROUP ONE TOOK A *PLACEBO*—A PILL IDENTICAL TO ASPIRIN, BUT CONTAINING NO ASPIRIN.

GROUP TWO RECEIVED ONE ASPIRIN A DAY.

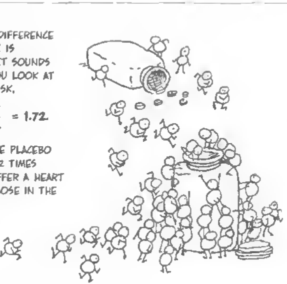
OVER A PERIOD AVERAGING NEARLY FIVE YEARS*, THE INVESTIGATORS RECORDED THE RESPONSES: HEART ATTACK OR NO HEART ATTACK. THE RESULT. (IN THE NUMBERS THAT FOLLOW, WE HAVE COMBINED FATAL AND NONFATAL HEART ATTACKS.)

	ATTACK	NO ATTACK	n	ATTACK RATE
PLACEBO	239	10,795	11,034	$\hat{p}_1 = \frac{239}{11{,}034} = .0217$
ASPIRIN	139	10,898	11,037	$\hat{p}_2 = \frac{139}{11{,}037} = .0126$

THE OBSERVED DIFFERENCE IN SUCCESS RATE IS $\hat{p}_1 - \hat{p}_2 = .0091$. IT SOUNDS SMALL UNTIL YOU LOOK AT THE RELATIVE RISK,

$$\frac{\hat{p}_1}{\hat{p}_2} = \frac{.0217}{.0126} = 1.72.$$

MEMBERS OF THE PLACEBO GROUP WERE 1.72 TIMES LIKELIER TO SUFFER A HEART ATTACK THAN THOSE IN THE ASPIRIN GROUP.



*THE STUDY WAS STOPPED EARLY BECAUSE OF ITS POSITIVE OUTCOME. IT WOULD HAVE BEEN UNWISE AND IMPRACTICAL TO DENY THE RESULTS TO THE GROUP TAKING THE PLACEBO.

The Model: THE PLACEBO AND ASPIRIN GROUP OBSERVATIONS ARE INDEPENDENT SAMPLES FROM TWO BINOMIAL POPULATIONS. FOR CONSISTENCY, WE REFER TO A HEART ATTACK AS A *SUCCESS (!)*.

PLACEBO
POPULATION ONE
CHANCE OF SUCCESS = p_1

ASPIRIN
POPULATION TWO
CHANCE OF SUCCESS = p_2

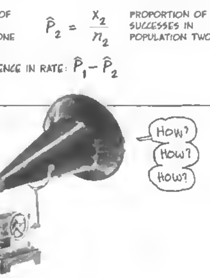
THE OBJECTIVE IS TO ESTIMATE THE TRUE DIFFERENCE, $p_1 - p_2$

FOR EACH POPULATION (ACTUALLY LARGE SAMPLES OF THE GENERAL POPULATION), WE HAVE THE FAMILIAR RANDOM VARIABLES

X_1	NUMBER OF SUCCESSES IN POPULATION ONE	X_2	NUMBER OF SUCCESSES IN POPULATION TWO
$\hat{P}_1 = \frac{X_1}{n_1}$	PROPORTION OF SUCCESSES IN POPULATION ONE	$\hat{P}_2 = \frac{X_2}{n_2}$	PROPORTION OF SUCCESSES IN POPULATION TWO

AND AN ESTIMATOR OF DIFFERENCE IN RATE: $\hat{P}_1 - \hat{P}_2$

AND NOW, LIKE A BROKEN RECORD, WE ASK OURSELVES, HOW IS $\hat{P}_1 - \hat{P}_2$ DISTRIBUTED?

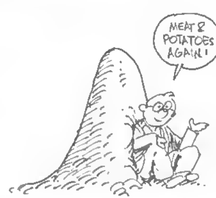


Sampling distribution for $\hat{P}_1 - \hat{P}_2$

FOR LARGE SAMPLES, $\hat{P}_1 - \hat{P}_2$ IS APPROXIMATELY NORMALLY DISTRIBUTED, MUCH AS IN THE CASE OF ONLY ONE SAMPLE. WE CAN MAKE THE USUAL Z-TRANSFORM TO GET A STANDARD NORMAL RANDOM VARIABLE (APPROXIMATELY)

$$z = \frac{\hat{P}_1 - \hat{P}_2 - (p_1 - p_2)}{\sigma(\hat{P}_1 - \hat{P}_2)}$$

BUT HOW DO WE FIND THAT STANDARD DEVIATION IN THE DENOMINATOR?



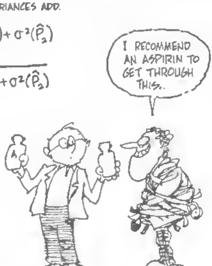
SINCE THE TWO SAMPLES ARE INDEPENDENT, SO ARE THE RANDOM VARIABLES $\hat{P}_1$ AND $\hat{P}_2$, AND THE TWO VARIANCES ADD.

$$\sigma^2(\hat{P}_1 - \hat{P}_2) = \sigma^2(\hat{P}_1) + \sigma^2(\hat{P}_2)$$

SO

$$\sigma(\hat{P}_1 - \hat{P}_2) = \sqrt{\sigma^2(\hat{P}_1) + \sigma^2(\hat{P}_2)}$$

AND NOW, KNOWING THE DISTRIBUTION OF THE TEST STATISTICS, WE CAN PROCEED TO ESTIMATE *CONFIDENCE INTERVALS* AND *TEST THE HYPOTHESIS* THAT ASPIRIN REDUCES HEART ATTACKS.



Confidence Intervals for $p_1 - p_2$

AS USUAL, THE CONFIDENCE INTERVALS FOR OUR ESTIMATE LOOK LIKE THIS:

$$p_1 - p_2 = \hat{p}_1 - \hat{p}_2 \pm z_{\frac{\alpha}{2}} SE(\hat{p}_1 - \hat{p}_2)$$

TRUE DIFFERENCE OF POPULATION PROPORTIONS — OBSERVED DIFFERENCE — CRITICAL VALUE — STANDARD ERROR

THE VARIANCES OF $\hat{P}_1$ AND $\hat{P}_2$ ADD, SO THE STANDARD ERROR BECOMES

$$SE(\hat{p}_1 - \hat{p}_2) = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

IN THE ASPIRIN STUDY, THE STANDARD ERROR IS

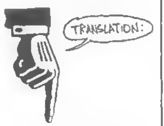
$$\sqrt{\frac{(.0217)(.9783)}{11{,}034} + \frac{(.0126)(.9874)}{11{,}037}}$$

$$= .00175$$

TO GET THE 95% CONFIDENCE INTERVAL FOR THE ASPIRIN STUDY, WE JUST PLUG IN THE OBSERVED VALUES:

$$p_1 - p_2 = .0091 \pm (1.96)(.00175)$$

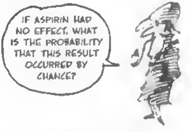
$$= .0091 \pm .0034$$



WE ARE AT LEAST 95% CONFIDENT THAT THE DIFFERENCE IN HEART ATTACK RATES IS BETWEEN .0057 AND .0125. DEFINITELY A POSITIVE NUMBER! WE ARE NOW AT LEAST 95% CONFIDENT THAT ASPIRIN REALLY DOES LOWER HEART ATTACK RATES.

hypothesis testing

THE FORMAL HYPOTHESIS-TESTING QUESTION IS



H_0, THE NULL HYPOTHESIS, IS THAT ASPIRIN HAD NO EFFECT. $p_1 = p_2$.

H_a, THE ALTERNATIVE, IS THAT ASPIRIN DOES REDUCE THE HEART ATTACK RATE. $p_1 > p_2$.

NOW WE NEED A TEST STATISTIC WITH A NORMAL DISTRIBUTION WHEN H_0 IS TRUE...

NOTE THAT UNDER H_0, THE TWO PROPORTIONS ARE THE SAME, $p_1 = p_2 = p$... SO LET'S POOL ALL THE DATA TO GET THE PROPORTION OF HEART ATTACKS IN *BOTH SAMPLES TOGETHER*:

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

WHEN THE NULL HYPOTHESIS IS TRUE, THE STANDARD ERROR DEPENDS ONLY ON THIS POOLED ESTIMATE:

$$SE_0(\hat{P}_1 - \hat{P}_2) = \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

AND WE CAN WRITE A TEST STATISTIC:

$$Z = \frac{\hat{P}_1 - \hat{P}_2}{SE_0(\hat{P}_1 - \hat{P}_2)}$$

(THE NUMERATOR WOULD ORDINARILY BE $\hat{P}_1 - \hat{P}_2 - (p_1 - p_2)$, BUT H_0 ASSUMES $p_1 - p_2 = 0$.)

FOR THE ASPIRIN STUDY, WE FIND

$$\hat{p} = \frac{378}{22{,}071}$$

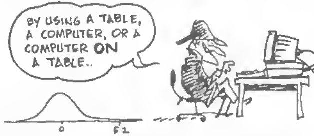
$$SE_0(\hat{P}_1 - \hat{P}_2) = .00175$$

SO

$$Z_{OBS} = \frac{.0091}{.00175} = 5.20$$

Z_{OBS} IS MORE THAN *FIVE STANDARD DEVIATIONS* FROM ZERO, A STRONG POSITIVE EFFECT USING A TABLE OR A COMPUTER, WE FIND THE P-VALUE.

$$\text{P-VALUE} = PR(Z \geq Z_{OBS}) = PR(Z \geq 5.2) = \mathbf{.0000001}$$



IF THE NULL HYPOTHESIS WERE TRUE, THE PROBABILITY OF OBSERVING AN EFFECT THIS LARGE IS *ONE IN TEN MILLION*—VERY STRONG EVIDENCE AGAINST H_0!!!

The general recipe:

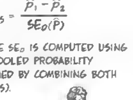
TO TEST THE NULL HYPOTHESIS

$$H_0: p_1 = p_2$$

COMPUTE THE TEST STATISTIC

$$Z_{OBS} = \frac{\hat{p}_1 - \hat{p}_2}{SE_0(\hat{P})}$$

(WHERE SE_0 IS COMPUTED USING THE POOLED PROBABILITY OBTAINED BY COMBINING BOTH GROUPS).

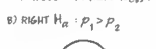


THE RELEVANT P-VALUE DEPENDS ON THE ALTERNATE HYPOTHESIS:

A) TWO-SIDED $H_a : p_1 \neq p_2$

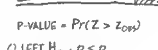
P-VALUE = $Pr(|Z| > |Z_{OBS}|)$

B) RIGHT $H_a : p_1 > p_2$



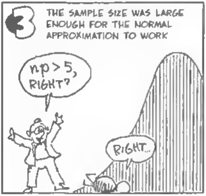
P-VALUE = $Pr(Z > Z_{OBS})$

C) LEFT $H_a : p_1 < p_2$



P-VALUE = $Pr(Z < Z_{OBS})$

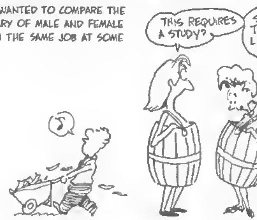
THE ANALYSIS OF THE ASPIRIN STUDY DEPENDED ON CERTAIN FEATURES OF THE EXPERIMENT DESIGNED TO ENSURE RANDOMNESS AND TO ELIMINATE BIAS:



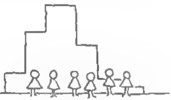
POINTS 1 AND 2 ARE ESSENTIAL PARTS OF MOST HUMAN CLINICAL TRIAL DESIGNS, BUT POINT 3 IS NOT ESSENTIAL. GOOD SMALL-SAMPLE TESTS DO EXIST AND ARE AVAILABLE IN COMPUTER SOFTWARE PACKAGES. THESE *NONPARAMETRIC* PROCEDURES DEPEND ON SIMPLE, BUT LENGTHY, PROBABILITY CALCULATIONS SIMILAR TO THE GAMBLING COMPUTATIONS WE ENCOUNTERED IN CHAPTER 4.

Comparing the MEANS of two populations

SUPPOSE WE WANTED TO COMPARE THE AVERAGE SALARY OF MALE AND FEMALE EMPLOYEES IN THE SAME JOB AT SOME COMPANY



POPULATION ONE IS THE WOMEN, AND POPULATION TWO IS THE MEN.



POPULATION ONE HAS MEAN SALARY μ_1 AND STANDARD DEVIATION σ_1

POPULATION TWO HAS MEAN SALARY μ_2 AND STANDARD DEVIATION σ_2

A RANDOM SAMPLE OF SIZE n_1 FROM GROUP 1 AND n_2 FROM GROUP 2 GIVES SAMPLE MEANS OF $\bar{x}_1$ AND $\bar{x}_2$ AND STANDARD DEVIATIONS s_1 AND s_1, RESPECTIVELY. THE ESTIMATOR OF $\mu_1 - \mu_2$ IS

$$\bar{X}_1 - \bar{X}_2$$

HOW GOOD AN ESTIMATOR IS $\bar{X}_1-\bar{X}_2$? FOR LARGE SAMPLE SIZES, IT'S APPROXIMATELY NORMAL (BY THE CENTRAL LIMIT THEOREM), AND ITS STANDARD ERROR IS

$$SE(\bar{X}_1-\bar{X}_2) = \sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}$$

(THE VARIANCES ADD, SINCE SAMPLES ARE INDEPENDENT.) NOW WE CAN PROCEED DIRECTLY TO:

confidence intervals:

FOR LARGE SAMPLE SIZES, THE $(1-\alpha)100\%$ CONFIDENCE INTERVAL FOR THE DIFFERENCE BETWEEN MEANS IS

$$\mu_1-\mu_2 = \bar{x}_1-\bar{x}_2 \pm z_{\frac{\alpha}{2}}\, SE(\bar{X}_1-\bar{X}_2)$$

hypothesis testing:

WE ASSESS THE NULL HYPOTHESIS THAT THE TWO POPULATION MEANS ARE EQUAL.

$$H_0: \ \mu_1 = \mu_2$$

THE TEST STATISTIC IS

$$Z_{OBS} = \frac{\bar{X}_1-\bar{X}_2}{SE(\bar{X}_1-\bar{X}_2)}$$

AND THE P-VALUES WORK IN THE USUAL WAY.

and how about comparing
SMALL SAMPLE
MEANS?

REMEMBER *CHAMELEON MOTORS?* THEIR COMPETITOR, *IGUANA AUTO*, CLAIMS THAT ITS STYROFOAM HOOD ORNAMENT GIVES BETTER FRONT END CRASH PROTECTION, AND THEY'VE CRASHED *SEVEN IGUANAS* TO PROVE IT!

THEIR RESULTS, COMPARED WITH CHAMELEON'S

CHAMELEON

1	\$150
2	\$400
3	\$720
4	\$500
5	\$930
n_1	5
$\bar{x}_1$	\$540
s_1	\$299

IGUANA

1	\$50
2	\$200
3	\$150
4	\$400
5	\$750
6	\$400
7	\$150
n_2	7
$\bar{x}_2$	\$300
s_2	\$238

THE t DISTRIBUTION CAN BE USED IF BOTH POPULATIONS ARE MOUND SHAPED AND HAVE THE SAME STANDARD DEVIATION $\sigma = \sigma_1 = \sigma_2$. THE ONLY WRINKLE IS THAT WE HAVE TO *POOL* THE SUM OF SQUARES ABOUT THE MEANS TO FORM A SINGLE ESTIMATE OF σ

$$s^2_{POOL} = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$$

THE STANDARD ERROR IS THE SAME AS FOR LARGE SAMPLES, EXCEPT THAT s_{POOL} REPLACES s_1 AND s_2:

$$SE(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{s^2_{POOL}}{n_1} + \frac{s^2_{POOL}}{n_2}}$$

$$= s_{POOL}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

THE $(1-\alpha)100\%$ CONFIDENCE INTERVAL IS

$$\mu_1 - \mu_2 = \bar{x}_1 - \bar{x}_2 \pm t_{\frac{\alpha}{2}}\, SE(\bar{X}_1 - \bar{X}_2)$$

WHERE $t_{\frac{\alpha}{2}}$ IS A CRITICAL VALUE OF t WITH $n_1 - n_2 - 2$ DEGREES OF FREEDOM

THE REPTILIAN CARMAKERS AGREE THAT THEIR STANDARD DEVIATIONS ARE CLOSE AND THEIR REPAIR HISTOGRAMS ARE MOUND-SHAPED THEY COMPUTE:

$$s_{POOL} = \sqrt{\frac{4\ 299^2 + 6\ 328^2}{10}} = 264$$

$$SE(\bar{X}_1 - \bar{X}_2) = 264\sqrt{\frac{1}{5} + \frac{1}{7}} = 154$$

THE 95% CONFIDENCE INTERVAL IS

$$\mu_1 - \mu_2 = 540 - 300 \pm t_{.025}(154)$$

$$= 240 \pm (2.23)(154)$$

$$= \mathbf{240 \pm 340}$$

SINCE THIS INCLUDES THE VALUE 0, IGUANA AUTOS HAS *NOT* SHOWN A SIGNIFICANT IMPROVEMENT IN REPAIR COSTS.

HERE'S AN EXAMPLE THAT SHOWS THE PITFALLS OF MINDLESSLY FOLLOWING THE COOKBOOK: A LARGE TAXI FLEET OWNER WANTS TO COMPARE THE GAS MILEAGE USING GAS A AND GAS B.

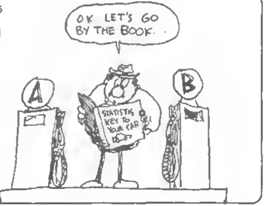
STARTING WITH 100 CABS, HE RANDOMLY ASSIGNS 50 TO EACH GASOLINE, AND, AFTER A DAY'S DRIVING, DETERMINES

	SAMPLE SIZE	MEAN MILEAGE	STANDARD DEVIATION
A	50	25	5.00
B	50	26	4.00

THE SAMPLE DIFFERENCE IS

$$\bar{x}_1 - \bar{x}_2 = 25 - 26 = -1$$

IS GAS B REALLY BETTER THAN GAS A?



OWING TO THE LARGE STANDARD DEVIATIONS, THE STANDARD ERROR IS PRETTY SUBSTANTIAL:

$$SE(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$= \sqrt{\frac{25}{50} + \frac{16}{50}}$$

$$= .905$$

AT THE 95% CONFIDENCE LEVEL, WE HAVE

$$\mu_1 - \mu_2 = \bar{x}_1 - \bar{x}_2 \pm z_{.025}(.905)$$

$$= -1 \pm (1.96)(.905)$$

$$= -1 \pm 1.774$$

THIS INCLUDES THE VALUE 0, CORRESPONDING TO $\mu_1 = \mu_2$

THE P-VALUE FOR THE ALTERNATE HYPOTHESIS, H_a: $\mu_1 \neq \mu_2$, IS

$$Pr(|Z| \geq |z_{obs}|) = Pr(|Z| \geq \frac{1}{.905})$$

$$= Pr(|Z| \geq 1.1) = 2(.1357)$$

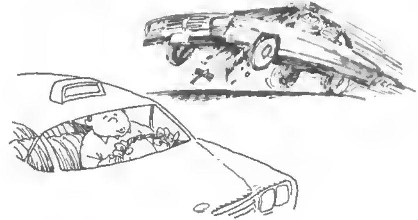
$$= .2714$$

THIS *EXCEEDS* THE $\alpha = .05$ SIGNIFICANCE LEVEL, SO WE CONCLUDE THAT THE EVIDENCE IN FAVOR OF EITHER GAS IS VERY WEAK.

PAIRED COMPARISONS
a better way to compare gasolines

THE TAXI OWNER FOLLOWED THE COOKBOOK EXACTLY HIS SAMPLES WERE RANDOM, AND HIS SAMPLE SIZE WAS LARGE ENOUGH. HE JUST FAILED TO *THINK* WHEN NECESSARY!

ALTHOUGH GAS B APPEARS TO BE SLIGHTLY BETTER THAN GAS A, THE CONFIDENCE INTERVAL WAS WIDE BECAUSE OF THE LARGE STANDARD DEVIATIONS—I.E., THE *MILEAGES VARIED WIDELY FROM ONE CAB TO THE NEXT.* WHY SUCH HIGH VARIABILITY? BECAUSE CABS—AND CABBIES—HAVE DIFFERENT *PERSONALITIES!*



A FAR BETTER WAY TO DO THIS STUDY IS TO ASSIGN GAS A AND GAS B TO THE *SAME CAB ON DIFFERENT DAYS.*

WE STILL RANDOMIZE THE TREATMENT BY FLIPPING A COIN TO DECIDE WHETHER TO USE GAS A ON TUESDAY OR WEDNESDAY. WE CAN ALSO CUT THE EXPERIMENT DOWN TO 10 CABS, SAVING THE OWNER A LOT OF TIME AND MONEY!

CAB	GAS A	GAS B	DIFFERENCE
1	27.01	26.95	0.06
2	20.00	20.44	−0.44
3	23.41	25.05	−1.64
4	25.22	26.32	−1.10
5	30.11	29.56	0.55
6	25.55	26.60	−1.05
7	22.23	22.93	−0.70
8	19.78	20.23	−0.45
9	33.45	33.95	−0.50
10	25.22	26.01	−0.79
MEAN	25.20	25.80	−0.60
STANDARD DEVIATION	4.27	4.10	0.61

NOTE THAT THE MEANS AND STANDARD DEVIATIONS OF GAS A AND GAS B ARE ABOUT THE SAME. THAT'S TO BE EXPECTED, SINCE THEY HAVE THE SAME SOURCE OF VARIABILITY AS IN THE UNPAIRED EXPERIMENT. BUT NOW THE *DIFFERENCE COLUMN* HAS A VERY *SMALL* STANDARD DEVIATION. THE DIFFERENCE COLUMN, BY COMPARING GAS PERFORMANCE WITHIN A SINGLE CAR, ELIMINATES VARIABILITY BETWEEN TAXIS.

THE DIFFERENCES d_i PROVIDE A *SINGLE MEASURE OF DIFFERENCE FOR EACH TAXI,* AND WE CAN USE IT TO MAKE A SMALL-SAMPLE t TEST STATISTIC

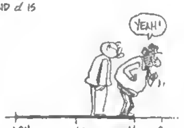
$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

THE 95% CONFIDENCE INTERVAL AROUND $\bar{d}$ IS

$$\mu_d = \underset{\text{SAMPLE MEAN}}{\bar{d}} \pm \underset{\text{CRITICAL VALUE}}{t_{.025}} \underset{\text{STANDARD ERROR}}{(s_d/\sqrt{n})}$$

$$= -.6 \pm (2.26)\left(\frac{.61}{\sqrt{10}}\right)$$

$$= -.60 \pm .44$$



SO WE HAVE $-1.04 \leq \mu_d \leq -.16$ WITH 95% CONFIDENCE, GOOD EVIDENCE THAT GAS B REALLY IS BETTER.

THE HYPOTHESIS-TESTING P-VALUE CAN BE FOUND USING A SOFTWARE PACKAGE:

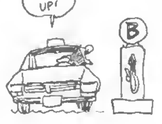
H_a: $\mu_d \neq 0$

$$\text{P-VALUE} = Pr(|t| \geq |t_{obs}|)$$

$$= Pr\left(|t| \geq \frac{.6}{.19}\right)$$

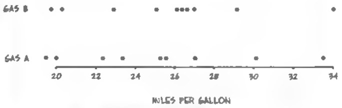
$$= Pr(|t| \geq 3.15)$$

$$= .012 < .05$$

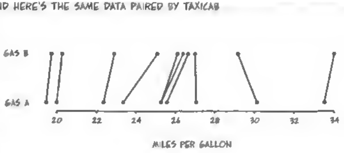


AGAIN, GAS B PASSES THE TEST.

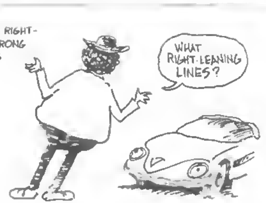
HERE ARE PLOTS OF THE GAS MILEAGE DATA. THE FIRST ONE SHOWS THE MILEAGES UNPAIRED:



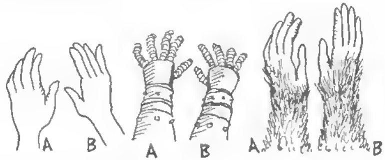
AND HERE'S THE SAME DATA PAIRED BY TAXICAB



THE PREDOMINANCE OF RIGHT-LEANING LINES IS A STRONG HINT THAT GAS B GIVES BETTER MILEAGE.



A PAIRED COMPARISON EXPERIMENT IS ONE OF THE MOST EFFECTIVE WAYS TO REDUCE NATURAL VARIABILITY WHILE COMPARING TREATMENTS. FOR EXAMPLE, IN COMPARING HAND CREAMS, THE TWO BRANDS ARE RANDOMLY ASSIGNED TO EACH SUBJECT'S RIGHT OR LEFT HANDS THIS ELIMINATES VARIABILITY DUE TO SKIN DIFFERENCES



OR, IN COMPARING TWO BREAKFAST CEREALS, EACH TASTER RATES BOTH CEREALS (IN RANDOM ORDER). THE PAIRED COMPARISON REMOVES THE NATURAL BIAS OF THE TASTER FOR OR AGAINST CEREAL IN GENERAL.

IN THIS CHAPTER, WE APPLIED THE BASIC IDEAS ABOUT CONFIDENCE INTERVALS AND HYPOTHESIS TESTING TO THE COMPARISON OF TWO POPULATIONS. THERE ARE INNUMERABLE FURTHER POSSIBILITIES WE COULD HAVE GONE ON TO DESCRIBE COMPARISONS OF:

- THE STANDARD DEVIATIONS OF TWO POPULATIONS WHEN SAMPLE SIZE IS SMALL,
- THE MEANS OF MORE THAN TWO POPULATIONS WHEN SAMPLE SIZE IS LARGE,
- THE MEANS OF MORE THAN TWO POPULATIONS WHEN SAMPLE SIZE IS SMALL,

ETC!

IN PRACTICE, STATISTICIANS DETERMINE THE GENERAL NATURE OF THE PROBLEM, AND THEN CONSULT THE RIGHT REFERENCE BOOK

THE ONLY THING REALLY NEW IN THE CHAPTER WAS THE IDEA OF THE *PAIRED COMPARISON TEST.* IN THE NEXT CHAPTER, WE'LL LOOK AT SOME OTHER KINDS OF EXPERIMENTAL DESIGNS


BUY A USED CHAMELEON?

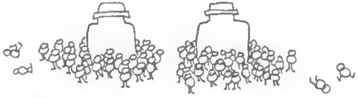
♦Chapter 10♦

EXPERIMENTAL DESIGN

THE DESIGN OF AN EXPERIMENT OFTEN SPELLS SUCCESS OR FAILURE. IN THE PAIRED COMPARISONS EXAMPLE, OUR STATISTICIAN CHANGED ROLES FROM PASSIVE NUMBER GATHERING AND ANALYSIS TO ACTIVE PARTICIPATION IN THE DESIGN OF THE EXPERIMENT.

IN THIS CHAPTER, WE INTRODUCE THE BASIC IDEAS OF EXPERIMENTAL DESIGN, WHILE LEAVING THE DETAILED NUMERICAL ANALYSIS TO YOUR HANDY STATISTICAL SOFTWARE PACK

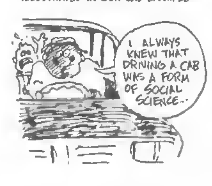
THE ELEMENTS OF A DESIGN ARE THE *EXPERIMENTAL UNITS* AND THE *TREATMENTS* THAT ARE TO BE ASSIGNED TO THE UNITS. THE OBJECTIVE OF ANY DESIGN IS TO COMPARE THE TREATMENTS.



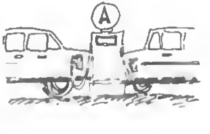
FOR MEDICAL TRIALS, THE *PATIENTS* ARE THE UNITS, AND THE *DRUGS* ARE THE TREATMENTS. IN THE MILEAGE EXAMPLE, THE EXPERIMENTAL UNITS ARE TAXICABS, AND THE TREATMENTS TO BE COMPARED ARE GAS A AND GAS B.

IN AGRICULTURAL EXPERIMENTS, THE EXPERIMENTAL UNITS ARE OFTEN PLOTS IN A FIELD, AND THE TREATMENTS MIGHT BE APPLICATION OF DIFFERENT WHEAT VARIETIES, PESTICIDES, FERTILIZERS, ETC.

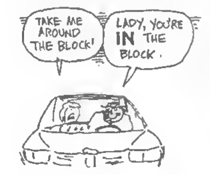
TODAY, EXPERIMENTAL DESIGN IDEAS ARE USED EXTENSIVELY IN *INDUSTRIAL PROCESS OPTIMIZATION, MEDICINE* AND *SOCIAL SCIENCE*. ANY EXPERIMENTAL DESIGN USES *THREE BASIC PRINCIPLES*, WHICH ARE CLEARLY ILLUSTRATED IN OUR CAB EXAMPLE



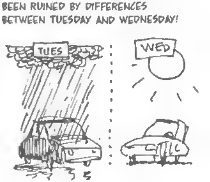
Replication: THE SAME TREATMENTS ARE ASSIGNED TO DIFFERENT EXPERIMENTAL UNITS WITHOUT REPLICATION, IT'S IMPOSSIBLE TO ASSESS NATURAL VARIABILITY AND MEASUREMENT ERROR



Local control REFERS TO ANY METHOD THAT ACCOUNTS FOR AND REDUCES NATURAL VARIABILITY. ONE WAY IS TO GROUP SIMILAR EXPERIMENTAL UNITS INTO *BLOCKS*. IN THE CAB EXAMPLE, BOTH GASOLINES WERE USED IN EACH CAR, AND WE SAY THAT THE CAB IS A BLOCK.

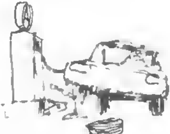


Randomization: THE ESSENTIAL STEP IN ALL STATISTICS! TREATMENTS MUST BE ASSIGNED RANDOMLY TO EXPERIMENTAL UNITS. FOR EACH TAXI, WE ASSIGNED GAS A TO TUESDAY OR WEDNESDAY BY FLIPPING A COIN. IF WE HADN'T, THE RESULTS COULD HAVE BEEN RUINED BY DIFFERENCES BETWEEN TUESDAY AND WEDNESDAY!



NOW SUPPOSE WE WANT TO INVESTIGATE THE EFFECT OF TWO BRANDS OF TIRES AS WELL AS TWO GASOLINES. WE HAVE FOUR POSSIBLE TREATMENTS, WHICH WE CAN LAY OUT IN A TWO-BY-TWO FACTORIAL DESIGN. THE TWO FACTORS ARE GAS AND TIRE MAKE.

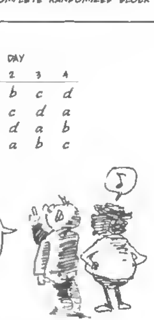
	GAS A	GAS B
TIRE A	a	b
TIRE B	c	d



WE CAN ASSIGN THE FOUR TREATMENTS AT RANDOM TO FOUR DIFFERENT DAYS FOR EACH CAB. ALL FOUR TREATMENTS (a, b, c, AND d) ARE REPEATED WITHIN EACH BLOCK (CAB). THIS IS CALLED A *COMPLETE RANDOMIZED BLOCK DESIGN.*

SO FAR, WE HAVE ASSUMED THAT EVERY DAY OF THE WEEK IS THE SAME, BUT WE CAN CONTROL FOR THIS, TOO, IN THE FOLLOWING WAY. USE ONLY *FOUR CABS,* AND ASSIGN THE TREATMENT ACCORDING TO THE TABLE AT RIGHT:

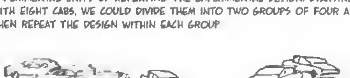
		DAY 1	2	3	4
CAB	1	a	b	c	d
	2	b	c	d	a
	3	c	d	a	b
	4	d	a	b	c



A FOUR-BY-FOUR TABLE WITH FOUR DIFFERENT ELEMENTS, EACH APPEARING ONCE IN EVERY COLUMN AND ROW, IS CALLED A **Latin square.** IN THIS EXPERIMENT, THE FOUR DAYS AND FOUR CABS GET ALL FOUR TREATMENTS EXACTLY ONCE.

THE RANDOMIZATION STEP PICKS A SINGLE LATIN SQUARE DESIGN AT RANDOM FROM A LIST OF ALL POSSIBLE FOUR-WAY LATIN SQUARES.

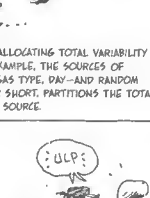
IF FOUR UNITS ISN'T ENOUGH, WE CAN INCREASE THE NUMBER OF EXPERIMENTAL UNITS BY *REPEATING* THE EXPERIMENTAL DESIGN. STARTING WITH EIGHT CABS, WE COULD DIVIDE THEM INTO TWO GROUPS OF FOUR AND THEN REPEAT THE DESIGN WITHIN EACH GROUP.



WE PROMISED NOT TO GO INTO THE DATA ANALYSIS IN ANY DETAIL, BUT HERE IS ROUGHLY HOW A COMPLEX DESIGN LIKE THIS IS HANDLED.

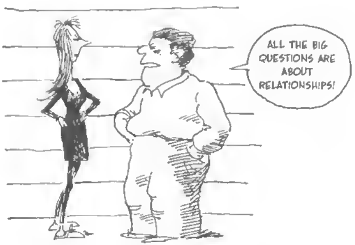
EXPERIMENTAL DESIGNS ARE ANALYZED BY ALLOCATING TOTAL VARIABILITY AMONG DIFFERENT SOURCES. IN THE CAB EXAMPLE, THE SOURCES OF VARIABILITY ARE THE CAB, THE TIRE MAKE, GAS TYPE, DAY—AND RANDOM ERROR. ANALYSIS OF VARIANCE, *ANOVA* FOR SHORT, PARTITIONS THE TOTAL VARIATION, ALLOCATING PORTIONS TO EACH SOURCE.

IN THE NEXT CHAPTER, WE EXPLAIN IN DETAIL ONE MODEL FOR ANALYZING COMPLEX DESIGNS: THE ***LINEAR REGRESSION MODEL***. IN LINEAR REGRESSION, YOU'LL BE ABLE TO SEE **ANOVA** UP CLOSE AND NUMERICAL...



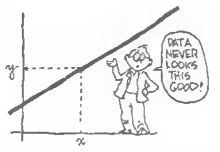
♦Chapter 11♦
REGRESSION

SO FAR, WE'VE DONE STATISTICS ON *ONE VARIABLE AT A TIME*, WHETHER IT CAME FROM A POPULATION OF PILL TAKERS, PICKLES, OR CRASHED CARS. IN THIS CHAPTER, WE'LL SEE HOW TO RELATE *TWO VARIABLES*: GIVEN THE ***WEIGHTS*** OF THE 92 STUDENTS IN CHAPTER 2, WE ASK HOW THEY ARE RELATED TO THE STUDENTS' *HEIGHTS*.

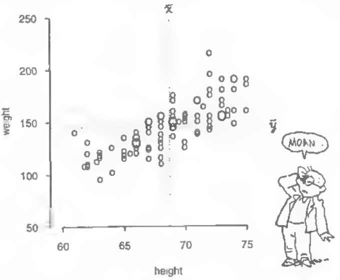


THIS IS AN EXAMPLE OF A BROAD CLASS OF IMPORTANT QUESTIONS: DOES ***BLOOD PRESSURE LEVEL*** PREDICT *LIFE EXPECTANCY?* DO *S.A.T. SCORES* PREDICT *COLLEGE PERFORMANCE?* DOES READING STATISTICS BOOKS MAKE YOU A *BETTER PERSON?*

IN MATH CLASS, YOU PROBABLY LEARNED TO SEE RELATIONSHIPS DISPLAYED AS *GRAPHS*. GIVEN x, YOU CAN PREDICT y. BUT IN STATISTICS, THINGS ARE NEVER SO CLEAN! WE KNOW (OR SUPPOSE WE KNOW) THAT HEIGHT HAS AN INFLUENCE ON WEIGHT— BUT IT'S NOT THE *SOLE* INFLUENCE. THERE ARE OTHER FACTORS, TOO, LIKE SEX, AGE, BODY TYPE, AND *RANDOM VARIATION*.



FOR THIS CHAPTER, LET'S LABEL THE WEIGHT DATA AS y AND THE HEIGHT DATA AS x. THUS (x_i, y_i) IS THE HEIGHT AND WEIGHT OF STUDENT i. WE DISPLAY THE POINTS (x_i, y_i) IN A 2-DIMENSIONAL DOT PLOT CALLED A *SCATTERPLOT*.



(SOME OF THE DOTS ARE BIGGER, BECAUSE THEY REPRESENT TWO OR THREE STUDENTS WITH THE SAME HEIGHT AND WEIGHT.)

CAN WE PREDICT A STUDENT'S WEIGHT y FROM HIS OR HER HEIGHT x?

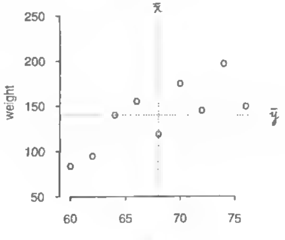
Regression analysis

FITS A STRAIGHT LINE TO THIS MESSY SCATTERPLOT. x IS CALLED THE *INDEPENDENT* OR *PREDICTOR* VARIABLE, AND y IS THE *DEPENDENT* OR *RESPONSE* VARIABLE. THE *REGRESSION* OR *PREDICTION* LINE HAS THE FORM

$$y = a+bx$$

TO ILLUSTRATE THE FITTING PROCESS, LET'S USE A SMALLER, RIGGED DATA SET WITH ONLY *NINE* STUDENT HEIGHT-WEIGHT PAIRS

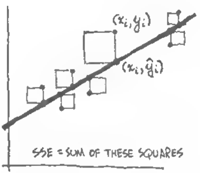
HEIGHT	WEIGHT
60	84
62	95
64	140
66	155
68	119
70	175
72	145
74	197
76	150



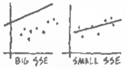
NOW HOW DO WE GET THE BEST-FITTING LINE?

THE IDEA IS TO *MINIMIZE* THE TOTAL SPREAD OF THE y VALUES FROM THE LINE. JUST AS WHEN WE DEFINED THE VARIANCE, WE LOOK AT ALL THE *SQUARED* y *DISTANCES* FROM THE LINE, AND ADD THEM UP TO GET THE *SUM OF SQUARED ERRORS:*

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$



IT'S AN AGGREGATE MEASURE OF HOW MUCH THE LINE'S "PREDICTED y_i," OR $\hat{y}_i$, DIFFER FROM THE ACTUAL DATA VALUES y_i.



The **regression** or **least squares** line

IS THE LINE WITH THE *SMALLEST SSE.*

HISTORICAL NOTE: WHY DO WE CALL THIS PROCEDURE *REGRESSION ANALYSIS?* AROUND THE TURN OF THE CENTURY, GENETICIST *FRANCIS GALTON* DISCOVERED A PHENOMENON CALLED *REGRESSION TOWARD THE MEAN.* SEEKING LAWS OF INHERITANCE, HE FOUND THAT SONS' HEIGHTS TENDED TO *REGRESS* TOWARD THE MEAN HEIGHT OF THE POPULATION, COMPARED TO THEIR FATHERS' HEIGHTS. TALL FATHERS TENDED TO HAVE SOMEWHAT SHORTER SONS, AND VICE VERSA. GALTON DEVELOPED *REGRESSION ANALYSIS* TO STUDY THIS EFFECT, WHICH HE OPTIMISTICALLY REFERRED TO AS "REGRESSION TOWARD MEDIOCRITY."

NOT TO BEAT AROUND THE BUSH, WE GIVE WITHOUT PROOF THE REGRESSION LINE'S FORMULA. IT'S MESSY BUT COMPUTABLE.

$$y = a + bx$$

WHERE

$$b = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

AND

$$a = \bar{y} - b\bar{x}$$

(HERE $\bar{x}$ AND $\bar{y}$ ARE THE MEANS OF $\{x_i\}$ AND $\{y_i\}$ RESPECTIVELY.)

BECAUSE SOME OF THESE EXPRESSIONS WILL SHOW UP AGAIN, WE ABBREVIATE THEM:

$$ss_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$ss_{yy} = \sum_{i=1}^{n}(y_i - \bar{y})^2$$

SUM OF SQUARES AROUND THE MEAN. THESE MEASURE THE SPREAD OF x_i AND y_i.

$$ss_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

THE CROSS PRODUCT DETERMINES (WITH ss_{xx}) THE COEFFICIENT b.

FOR THE RIGGED DATA, HERE'S THE WHOLE COMPUTATION:

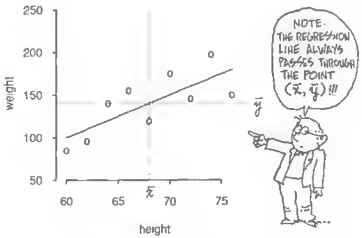
x_i	y_i	$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
60	84	-8	-56	64	3136	448
62	95	-6	-45	36	2025	270
64	140	-4	0	16	0	0
66	155	-2	15	4	225	-30
68	119	0	-21	0	441	0
70	175	2	35	4	1225	70
72	145	4	5	16	25	20
74	197	6	57	36	3249	342
76	150	8	10	64	100	80
SUM=612	1260			$SS_{xx}=240$	$SS_{yy}=10426$	$SS_{xy}=1200$

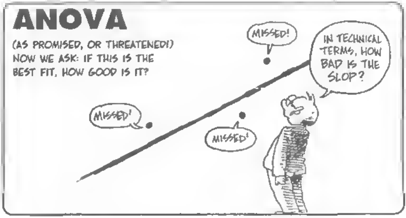
$\bar{x}=68$ $\bar{y}=140$

WHICH GIVES VALUES OF a AND b

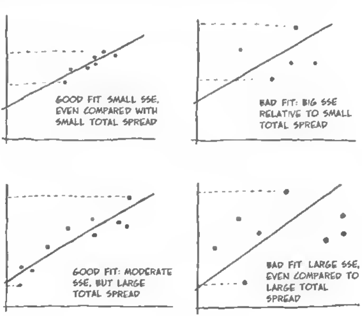
$$b=\frac{1200}{240}=5 \qquad a=\bar{y}-b\bar{x}=140-5(68)=-200$$

SO $y=-200+5x$





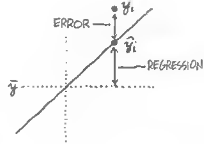
AS YOU CAN IMAGINE, THE ANSWER TO THIS QUESTION DEPENDS ON HOW SLOPPILY THE DATA POINTS ARE SPREAD OUT, I.E., HOW BIG *SSE* IS, RELATIVE TO THE TOTAL SPREAD OF THE DATA SOME EXAMPLES.



LET'S QUANTIFY THIS BY APPORTIONING THE VARIABILITY IN y. REFER TO THE PICTURE AT RIGHT FOR GUIDANCE. WE LET

$$\hat{y}_i = a + bx_i$$

THUS, $\hat{y}_i$ ARE THE PREDICTED WEIGHTS DETERMINED BY THE REGRESSION LINE.



ANOVA table

SOURCE OF VARIABILITY	SUM OF SQUARES	VALUE FOR RIGGED DATA
REGRESSION	$SSR = \sum_{1}^{n} (\hat{y}_i - \bar{y})^2$	6000
ERROR	$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$	4426
TOTAL	$SS_{yy} = \sum_{i=1}^{n} (y_i - \bar{y})^2$	10,426

(BY THE WAY, IT IS NOT OBVIOUS THAT $SS_{yy} = SSR + SSE$—BUT IT'S TRUE!) ANYWAY, HERE IS HOW THE REGRESSION AND ERROR SUMS OF SQUARES ARE CALCULATED FOR THE RIGGED DATA SET, WITH $y = -200 + 5x$.

			REGRESSION		ERROR	
x_i	y_i	$\hat{y}_i$	$(\hat{y}_i - \bar{y})$	$(\hat{y}_i - \bar{y})^2$	$(y_i - \hat{y}_i)$	$(y_i - \hat{y}_i)^2$
60	94	100	-40	1600	-16	256
62	95	110	-30	900	-15	225
64	140	120	-20	400	20	400
66	155	130	-10	100	25	625
68	119	140	0	0	-21	441
70	175	150	10	100	25	625
72	145	160	20	400	-15	225
74	197	170	30	900	27	729
76	150	180	40	1600	-30	900
$\bar{x} = 68$	$\bar{y} = 140$			SSR = 6000		SSE = 4426

SSR MEASURES THE TOTAL VARIABILITY DUE TO THE REGRESSION, I.E., THE PREDICTED VALUES OF y. *SSE* WE'VE ALREADY MET. NOTE THAT

$$\frac{SSE}{SS_{yy}}$$

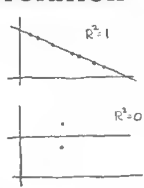
IS THE PROPORTION OF ERROR, RELATIVE TO THE TOTAL SPREAD

The squared correlation

IS THE PROPORTION OF THE TOTAL SS_{yy} ACCOUNTED FOR BY THE REGRESSION:

$$R^2 = \frac{SSR}{SS_{yy}} = 1 - \frac{SSE}{SS_{yy}}$$

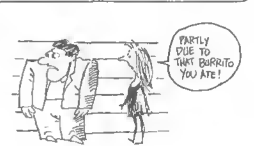
(BECAUSE $SSR = SS_{yy} - SSE$). R^2 IS ALWAYS LESS THAN 1. THE CLOSER IT IS TO 1, THE TIGHTER THE FIT OF THE CURVE. $R^2 = 1$ CORRESPONDS TO PERFECT FIT.



CALCULATING R^2 FOR THE RIGGED DATA SET, WE GET

$$R^2 = \frac{6000}{10{,}426} = .58$$

58% OF THE VARIATION IN WEIGHT IS EXPLAINED BY HEIGHT. THE OTHER 42% IS "ERROR"



ALTERNATELY, THE

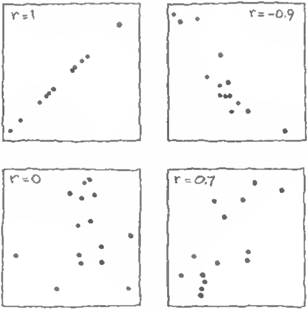
correlation coefficient

IS THE SQUARE ROOT OF R^2 WITH THE SIGN OF b.

$$r = (\text{SIGN OF } b)\sqrt{R^2}$$

THUS, r IS + IF THE LINE GOES UP TO THE RIGHT AND − IF IT GOES DOWN TO THE RIGHT.

r MEASURES THE TIGHTNESS OF FIT, AS WELL AS SAYING WHETHER INCREASING x MAKES y GO UP OR DOWN.

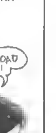


NOW LET'S BE HONEST: NOBODY—WELL, *ALMOST* NOBODY—DOES THESE CALCULATIONS BY HAND ANYMORE. WITH A COMPUTER, ALL THIS WORK CAN BE DONE IN *ONE LINE OF CODE*...

USING THE *MINITAB* STATISTICAL SOFTWARE SYSTEM, DEVELOPED AT PENN STATE, THE SINGLE COMMAND LOOKS LIKE THIS.

```
MTB > regress 'weight' on 1 independent variable 'height'
```

AND THE RESULTS ARE




```
The regression equation is

WEIGHT = - 200 + 5.00 height

Predictor  Coef     Stdev   t-ratio    p
Constant   -200.0   110.7   -1.81      0.114
height     5.000    1.623    3.08      0.018

s = 25.15    R-sq = 57.5% R-sq(adj) = 51.5%


Analysis of Variance

SOURCE       DF    SS       MS       F      p
Regression   1     6000.0   6000.0   9.49   0.018
Error        7     4426.0   632.3
Total        8     10426.0
```

NOW LET'S DO IT TO THE *REAL* DATA OF 92 STUDENTS:

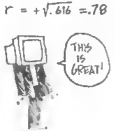
```
MTB > regress 'weight' on 1 independent variable 'height'
```

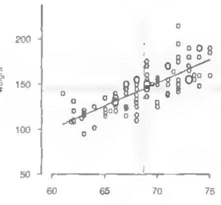
AND THE RESULTS

```
The regression equation is
WEIGHT = - 205 + 5.09 HEIGHT

Predictor   Coef      Stdev    t-ratio    p
Constant   -204.74    29.16    -7.02      0.000
height      5.0918    0.4237   12.02      0.000

s = 14.79    R-sq = 61.6%  R-sq(adj) = 61.2%


Analysis of Variance

SOURCE       DF      SS       MS        F        p
Regression   1       31592    31592     144.38   0.000
Error        90      19692    219
Total        91      51284
```

HERE IS THE SCATTERPLOT WITH THE FITTED REGRESSION LINE. THE CORRELATION COEFFICIENT FOR THIS DATA SET IS

$r = +\sqrt{.616} = .78$



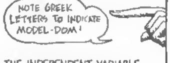


STATISTICAL INFERENCE

UP TO NOW, WE HAVE BEEN DOING *DATA ANALYSIS*, DESCRIBING THE NEAREST LINEAR RELATIONSHIP BETWEEN THE OBSERVED DATA x AND y. NOW LET'S SHIFT OUR POINT OF VIEW, AND REGARD THE 92 STUDENTS AS A *SAMPLE* OF THE POPULATION OF STUDENTS AT LARGE. WHAT CAN WE INFER?

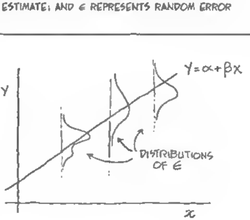
A ***REGRESSION MODEL*** FOR THE WHOLE POPULATION IS A LINEAR RELATIONSHIP

$$Y = \alpha + \beta x + \epsilon$$

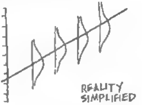


Y IS THE DEPENDENT RANDOM VARIABLE; x IS THE INDEPENDENT VARIABLE (WHICH MAY OR MAY NOT BE RANDOM); α AND β ARE THE UNKNOWN PARAMETERS WE SEEK TO ESTIMATE; AND ϵ REPRESENTS RANDOM ERROR FLUCTUATIONS.

FOR THE HEIGHT VS. WEIGHT MODEL, Y IS WEIGHT, x IS HEIGHT, α AND β ARE UNKNOWN, AND YOU CAN THINK OF ϵ AS THE ***RANDOM COMPONENT*** OF THE WEIGHTS Y FOR EACH VALUE OF HEIGHT x.



THE DISTRIBUTION OF ϵ IS IN FACT *DIFFERENT* FOR DIFFERENT VALUES OF x: 5-FOOTERS VARY LESS IN THEIR WEIGHT THAN 6-FOOTERS. NEVERTHELESS, WE NOW MAKE A SIMPLIFYING ASSUMPTION: LET'S SUPPOSE THAT FOR ALL VALUES OF x, THE ϵ'S ARE *INDEPENDENT, NORMAL,* AND HAVE THE *SAME STANDARD DEVIATION* $\sigma = \sigma(\epsilon)$ *AND MEAN* $\mu = 0$



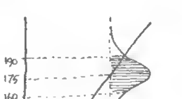
SO... MAYBE THE WEIGHT MODEL MIGHT BE

$$Y = -125 + 4x + \epsilon$$

ϵ IS NORMAL WITH $\mu = 0$ AND $\sigma = 15$ POUNDS (SAY). THEN, ACCORDING TO THIS MODEL, STUDENTS WHO ARE 6'4" (76 INCHES) HAVE THE DISTRIBUTION OF

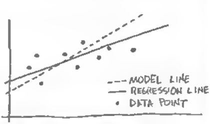
$$Y = -125 + 4(76) + \epsilon$$
$$= 175 + \epsilon$$

SO, FOR $x = 76$, Y IS NORMAL WITH MEAN 175 AND STANDARD DEVIATION 15 POUNDS.

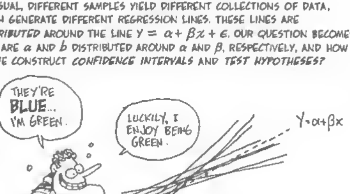


NOW, GIVEN THE MODEL $Y = \alpha + \beta x + \epsilon$, WE WANT TO DO AS WE'VE DONE REPEATEDLY IN THE LAST FEW CHAPTERS TAKE A *SAMPLE* AND USE IT TO *ESTIMATE* α AND β.

ONE CAN SHOW THAT THE a AND b WE GOT BY THE LEAST-SQUARES METHOD ARE **BLUE**: THE BEST LINEAR UNBIASED ESTIMATORS OF α AND β (WHATEVER THAT MEANS!).



AS USUAL, DIFFERENT SAMPLES YIELD DIFFERENT COLLECTIONS OF DATA, WHICH GENERATE DIFFERENT REGRESSION LINES. THESE LINES ARE *DISTRIBUTED* AROUND THE LINE $Y = \alpha + \beta x + \epsilon$. OUR QUESTION BECOMES HOW ARE a AND b DISTRIBUTED AROUND α AND β, RESPECTIVELY, AND HOW DO WE CONSTRUCT *CONFIDENCE INTERVALS* AND *TEST HYPOTHESES?*

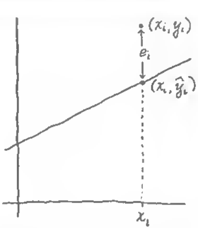


FOR EACH DATA POINT (x_i, y_i), WE HAVE

$$y_i = a + bx_i + e_i$$

WHERE $e_i = y_i - \hat{y}_i$ IS THE y-DISTANCE OF y_i FROM THE REGRESSION LINE THE e_i ARE *SAMPLE VALUES OF* ϵ, AND THEY GIVE US AN ESTIMATOR s FOR $\sigma(\epsilon)$:

$$s = \sqrt{\frac{\sum_{i=1}^{n} e_i^2}{n-2}}$$

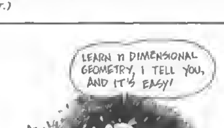


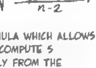
(WHY $n-2$ IN THE DENOMINATOR? BECAUSE WE HAVE USED UP TWO DEGREES OF FREEDOM TO COMPUTE a AND b, LEAVING $n-2$ INDEPENDENT PIECES OF INFORMATION TO ESTIMATE σ.)

ALTHOUGH IT ISN'T OBVIOUS, WE CAN ALSO WRITE s AS

$$s = \sqrt{\frac{SS_{yy} - bSS_{xy}}{n-2}}$$

A FORMULA WHICH ALLOWS US TO COMPUTE s DIRECTLY FROM THE SAMPLE STATISTICS.





TO REPEAT, s IS AN ESTIMATOR OF *HOW WIDELY THE DATA POINTS WILL BE SCATTERED AROUND THE LINE.*

confidence intervals

THE 95% CONFIDENCE INTERVALS FOR α AND β HAVE THAT OLD, FAMILIAR FORM:

$$\beta = b \pm t_{.025} SE(b)$$

$$\alpha = a \pm t_{.025} SE(a)$$

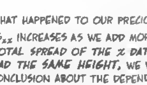
WHERE WE USE THE t DISTRIBUTION WITH $n-2$ DEGREES OF FREEDOM (FOR THE SAME REASON AS ABOVE).

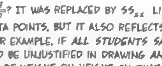
THE STANDARD ERRORS, HOWEVER, LOOK RATHER UNFAMILIAR. THEY ARE (WITHOUT DERIVATION):

$$SE(b) = \frac{s}{\sqrt{SS_{xx}}}$$

$$SE(a) = s\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{SS_{xx}}}$$

WHAT HAPPENED TO OUR PRECIOUS $\frac{1}{\sqrt{n}}$? IT WAS REPLACED BY SS_{xx}. LIKE n, SS_{xx} INCREASES AS WE ADD MORE DATA POINTS, BUT IT ALSO REFLECTS THE *TOTAL SPREAD OF THE x DATA.* FOR EXAMPLE, IF *ALL STUDENTS SAMPLED HAD THE SAME HEIGHT,* WE WOULD BE UNJUSTIFIED IN DRAWING ANY CONCLUSION ABOUT THE DEPENDENCE OF WEIGHT ON HEIGHT. IN THAT CASE, $SS_{xx} = 0$, GIVING $b = \infty$ AND *INFINITELY WIDE* CONFIDENCE INTERVALS.





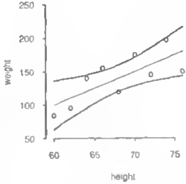
MORE QUESTIONS:

HOW WELL CAN WE PREDICT THE ***MEAN RESPONSE*** Y AT A FIXED VALUE x_0? FOR INSTANCE, WHAT IS THE MEAN WEIGHT OF STUDENTS OF HEIGHT 76 INCHES? THE 95% CONFIDENCE INTERVAL FOR $Y = \alpha + \beta x_0$ IS

$$\alpha + \beta x_0 = a + b x_0 \pm t_{.025} SE(\hat{y})$$

WHERE

$$SE(\hat{y}) = s\sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{SS_{xx}}}$$



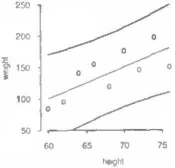
SUPPOSE A NEW STUDENT ENROLLS, WHO HAS HEIGHT x_{NEW}. HOW WELL CAN WE PREDICT Y_{NEW} WITHOUT MEASURING IT?

THE 95% PREDICTION INTERVAL FOR A NEW INDIVIDUAL Y_{NEW} WITH OBSERVED x_{NEW} IS

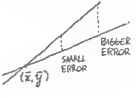
$$Y_{NEW} = a + b x_{NEW} \pm t_{.025} SE(Y_{NEW})$$

WHERE

$$SE(Y_{NEW}) = s\sqrt{1 + \frac{1}{n} + \frac{(x_{NEW} - \bar{x})^2}{SS_{xx}}}$$



BOTH THESE STANDARD ERRORS CONTAIN A TERM THAT GROWS LARGER AS THE x-VALUE, x_0 OR x_{NEW}, GETS FARTHER FROM THE MEAN VALUE $\bar{x}$. WHY DOES THE ERROR INCREASE FARTHER FROM $\bar{x}$? BECAUSE, IF YOU WIGGLE THE REGRESSION LINE, IT MAKES MORE OF A DIFFERENCE FARTHER FROM THE MEAN! (REMEMBER, THE LINE ALWAYS PASSES THROUGH $(\bar{x}, \bar{y})$.)

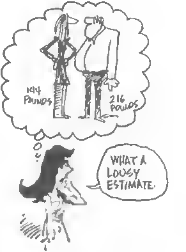


LET'S WORK IT OUT FOR THE RIGGED DATA: FOR THE *MEAN WEIGHT* WHEN $x = 76$ INCHES, WE HAVE $b = -200$ AND $a = 5$. THEN

$$Y = -200 + 5(76) \pm (2.365)(25.15)$$

$$= 180 \pm (2.365)(25.15)\sqrt{.3777}$$

$$= \mathbf{180 \pm 36.34} \text{ POUNDS}$$



THE ESTIMATED MEAN OF 6'4" STUDENTS IS 180 POUNDS, AND WE'RE 95% CONFIDENT THAT WE'RE WITHIN ***36 POUNDS*** OF THE TRUE MEAN.

FOR A *NEW* STUDENT WHO'S 6'4", WE USE OUR RIGGED SAMPLE OF NINE POINTS TO PREDICT THAT

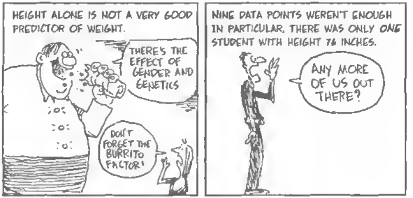
$$Y_{NEW} = -200 + 5(76) \pm (2.365)(25.15)\sqrt{1 + \frac{1}{9} + \frac{(76-68)^2}{240}}$$

$$= 180 \pm (2.365)(29.51)$$

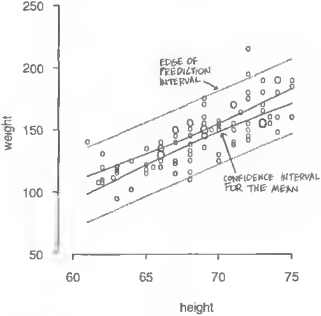
$$= \mathbf{180 \pm 70} \text{ POUNDS}$$

WE TELL THE FOOTBALL COACH THAT WE'RE PRETTY SURE THE NEW GUY WEIGHS SOMEWHERE BETWEEN ***110*** AND ***250!!!***

THE INTERVALS ARE PRETTY TERRIBLE! WHAT'S THE PROBLEM? THERE ARE TWO PROBLEMS, ACTUALLY:

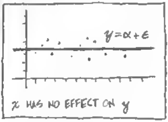


THE PENN STATE STUDENTS GIVE BETTER ESTIMATES.



hypothesis testing

THE COMPLETE SKEPTIC MIGHT SUGGEST THAT THERE IS *NO RELATIONSHIP* BETWEEN HEIGHT AND WEIGHT. THIS AMOUNTS TO SAYING THAT $\beta = 0$.



WE TAKE THIS AS THE *NULL HYPOTHESIS.*

$$H_0 : \beta = 0$$

IN THAT CASE, THE TEST STATISTIC

$$t = \frac{b}{SE(b)}$$

HAS THE t DISTRIBUTION WITH $n-2$ DEGREES OF FREEDOM. AS USUAL, THE SIGNIFICANCE TEST DEPENDS ON THE ALTERNATE HYPOTHESIS.

$t > t_\alpha$ FOR $H_a : \beta > 0$

$t < t_\alpha$ FOR $H_a : \beta < 0$

$|t| > |t_{\alpha/2}|$ FOR $H_a : \beta \neq 0$

FOR THE RIGGED WEIGHT DATA, WE STRONGLY SUSPECT THE ALTERNATE HYPOTHESIS SHOULD BE

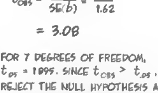
$$H_a : \beta > 0$$

WE TEST

$$t_{OBS} = \frac{5}{SE(b)} = \frac{5}{1.62}$$

$$= 3.08$$

FOR 7 DEGREES OF FREEDOM, $t_{.05} = 1895$. SINCE $t_{OBS} > t_{.05}$, WE REJECT THE NULL HYPOTHESIS AT THE $\alpha = 05$ SIGNIFICANCE LEVEL AND CONCLUDE THAT THERE IS A SIGNIFICANT, POSITIVE RELATIONSHIP BETWEEN HEIGHT AND WEIGHT.



Multiple linear regression

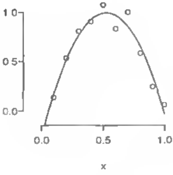
WE CAN USE THE SAME BASIC IDEAS TO ANALYZE RELATIONSHIPS BETWEEN A DEPENDENT VARIABLE AND *SEVERAL* INDEPENDENT VARIABLES:

$$Y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \ldots \beta_n x_n + \epsilon$$

FOR EXAMPLE, WEIGHT IS DETERMINED BY A NUMBER OF FACTORS OTHER THAN HEIGHT: AGE, SEX, DIET, BODY TYPE, ETC.

MATRIX ALGEBRA AND A COMPUTER COMBINE TO MAKE SUCH PROBLEMS EASY TO ANALYZE.

Non-linear regression



SOMETIMES DATA OBVIOUSLY FIT A *NON-LINEAR* CURVE. STATISTICIANS HAVE A BAG OF TRICKS FOR USING *LINEAR* REGRESSION TECHNIQUES FOR NON-LINEAR PROBLEMS. THE SIMPLEST OF THESE IS TO WRITE Y AS A *POLYNOMIAL*

$$Y = \alpha + \beta_1 x + \beta_2 x^2 + \epsilon$$

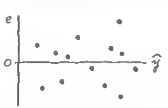
AND TREAT x AND x^2 AS INDEPENDENT VARIABLES IN A LINEAR MODEL.

Regression diagnostics

FITTING A COMPLEX MODEL TO DATA CAN SOMETIMES OBSCURE MANY DIFFICULTIES. WE USE REGRESSION DIAGNOSTIC PROCEDURES TO UNCOVER ANY LURKING NASTY SURPRISES.

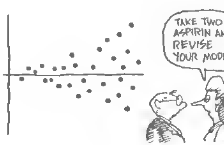
THE SIMPLEST PROCEDURE IS TO PLOT THE ***RESIDUALS*** e_i AGAINST THE ***PREDICTOR*** $\hat{y}_i$. REMEMBER, THE ERROR ϵ IS ASSUMED TO BE INDEPENDENT OF x

A ***RANDOM*** SCATTERPLOT INDICATES THAT THE MODEL ASSUMPTIONS ARE PROBABLY OK.



ANY ***PATTERN*** INDICATES A DEFINITE PROBLEM WITH THE MODEL ASSUMPTIONS.

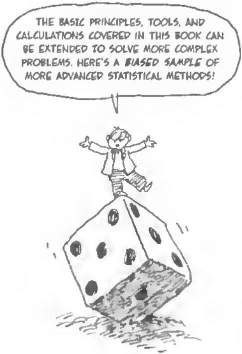
A TYPICAL LURKING NASTY SURPRISE (WHICH EXISTS IN THE HEIGHT/WEIGHT DATA) IS THAT ERRORS ARE ***HETEROSCEDASTIC***: I.E. THE SPREAD OF e INCREASES AS y INCREASES.



IN THIS CHAPTER, WE HAVE SUMMARIZED THE BASIC IDEAS AND TECHNIQUES OF REGRESSION ANALYSIS, THE STUDY OF STATISTICAL RELATIONSHIPS BETWEEN VARIABLES. THIS CONCLUDES OUR DETAILED DISCUSSION OF BASIC STATISTICAL METHODS. IN OUR FINAL CHAPTER, WE'LL BRIEFLY REVIEW A FEW REMAINING TOPICS AND ISSUES.

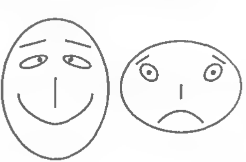
♦Chapter 12♦

CONCLUSION



DATA DISPLAY

WE SAW HOW TO DISPLAY *ONE* VARIABLE WITH A DOT PLOT AND *TWO* VARIABLES USING A SCATTERPLOT—BUT HOW DO WE GRAPHICALLY DISPLAY *MORE THAN TWO* VARIABLES ON A FLAT PAGE? AMONG THE MANY POSSIBILITIES, A CARTOON GUIDE HAS TO MENTION *HERMAN CHERNOFF'S* SIMPLE IDEA: USING THE HUMAN FACE, ASSIGN EACH FEATURE TO A VARIABLE AND DRAW THE RESULTING *CHERNOFF FACES*:



x = EYEBROW SLANT
y = EYE SIZE
z = NOSE LENGTH
t = MOUTH LENGTH
β = FACE HEIGHT
ETC...

Statistical analysis of
MULTIVARIATE DATA

AN ASSORTMENT OF MULTIVARIATE MODELS HELP TO ANALYZE AND DISPLAY n-DIMENSIONAL DATA. SOME MULTIVARIATE TECHNIQUES:

Cluster analysis

SEEKS TO DIVIDE THE POPULATION INTO HOMOGENEOUS SUBGROUPS. FOR EXAMPLE, BY ANALYZING CONGRESSIONAL VOTING PATTERNS, WE FIND THAT REPRESENTATIVES FROM THE *SOUTH* AND *WEST* FORM TWO DISTINCT CLUSTERS

Discriminant analysis

IS THE REVERSE PROCESS. FOR EXAMPLE, A COLLEGE ADMISSIONS OFFICE MIGHT LIKE TO FIND DATA GIVING ADVANCE WARNING WHETHER AN APPLICANT WILL GO ON TO BE A *SUCCESSFUL GRADUATE* (DONATES HEAVILY TO THE ALUMNI FUND) OR AN *UNSUCCESSFUL* ONE (GOES OUT TO DO GOOD IN THE WORLD AND IS NEVER HEARD FROM AGAIN).

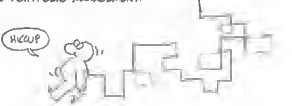
Factor analysis

SEEKS TO EXPLAIN HIGH-DIMENSIONAL DATA WITH A SMALLER NUMBER OF VARIABLES. A PSYCHOLOGIST MAY GIVE A TEST WITH 100 QUESTIONS, WHILE SECRETLY ASSUMING THAT THE ANSWERS DEPEND ON ONLY A FEW *FACTORS*: EXTROVERSION, AUTHORITARIANISM, ALTRUISM, ETC. THE TEST RESULTS WOULD THEN BE SUMMARIZED USING ONLY A FEW COMPOSITE SCORES IN THOSE DIMENSIONS.

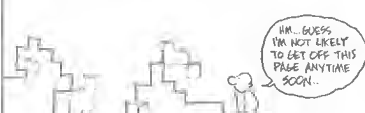
THERE IS ALSO MORE TO

PROBABILITY:

Random walks BEGIN WITH A COIN FLIP. SUPPOSE YOU MOVE AHEAD ONE STEP FOR A HEAD AND BACK ONE STEP FOR A TAIL. (USING TWO COINS, YOU CAN DO THIS IN TWO DIMENSIONS.) REPEATED FLIPS PRODUCE A STOCHASTIC PROCESS CALLED A *RANDOM WALK*. RANDOM WALK MODELS ARE USED IN *STOCK OPTION TRADING* AND *PORTFOLIO MANAGEMENT*.



Time series analysis DEALS WITH DATA SETS, WHICH, LIKE THE RANDOM WALK, *ACCUMULATE OVER TIME*: LOCAL AND GLOBAL TEMPERATURES, THE PRICE OF OIL, ETC. IN *TIME SERIES ANALYSIS*, RANDOM MODELS ARE USED TO FORECAST FUTURE VALUES.



WE'VE ALREADY SEEN HOW THE *COMPUTER* HELPS WITH ANALYSIS AND ARITHMETIC. THERE ARE ALSO SOME STATISTICAL IDEAS THAT OWE THEIR VERY *EXISTENCE* TO THE COMPUTER:

Image analysis

A COMPUTER IMAGE MIGHT CONSIST OF 1000 BY 1000 PIXELS, WITH EACH DATA POINT REPRESENTED FROM A RANGE OF 16.7 MILLION COLORS AT ANY PIXEL. STATISTICAL IMAGE ANALYSIS SEEKS TO EXTRACT MEANING FROM "INFORMATION" LIKE THIS.

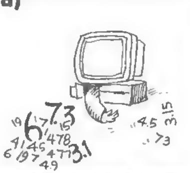
Resampling

SOMETIMES, STANDARD ERRORS AND CONFIDENCE LIMITS ARE IMPOSSIBLE TO FIND. ENTER *RESAMPLING*, A TECHNIQUE THAT TREATS THE SAMPLE *AS THOUGH IT WERE THE POPULATION*. THESE TECHNIQUES GO BY SUCH NAMES AS *RANDOMIZATION*, *JACKKNIFE*, AND *BOOTSTRAPPING*.

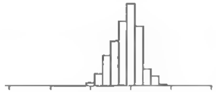
resampling (cont'd)

TO DO RESAMPLING, THE COMPUTER

* RESAMPLES THE SAMPLE
* COMPUTES THE ESTIMATE FOR THE RESAMPLE
* REPEATS THE FIRST TWO STEPS MANY TIMES, FINDING THE SPREAD OF THE RESAMPLED ESTIMATES



REMEMBER THE CORRELATION COEFFICIENT r OF THE 92 HEIGHT-WEIGHT PAIRS OF CHAPTER 11? WHAT'S THE *STANDARD ERROR OF r?* THE COMPUTER TAKES 200 BOOTSTRAP SAMPLES FROM THE 92 DATA POINTS, COMPUTES r EACH TIME, AND PLOTS A HISTOGRAM OF THE r VALUES



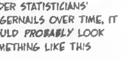
NOTE THAT THE SPREAD OF THE BOOTSTRAP ESTIMATES IS RELATIVELY SMALL.

DATA QUALITY

SEEMINGLY SMALL ERRORS IN SAMPLING, MEASUREMENT, AND DATA RECORDING CAN PLAY HAVOC WITH ANY ANALYSIS. *R. A. FISHER*, GENETICIST AND FOUNDER OF MODERN STATISTICS, NOT ONLY DESIGNED AND ANALYZED ANIMAL BREEDING EXPERIMENTS, HE ALSO *CLEANED THE CAGES* AND *TENDED THE ANIMALS*, BECAUSE HE KNEW THAT THE LOSS OF AN ANIMAL WOULD INFLUENCE HIS RESULTS

MODERN STATISTICIANS, WITH THEIR COMPUTERS, DATABASES, AND GOVERNMENT GRANTS, HAVE LOST SOME OF THIS HANDS-ON ATTITUDE.

IF YOU GRAPHED THE MEAN MASS OF RAT DROPPINGS UNDER STATISTICIANS' FINGERNAILS OVER TIME, IT WOULD *PROBABLY* LOOK SOMETHING LIKE THIS



Innovation

THE BEST SOLUTIONS ARE NOT ALWAYS IN THE BOOK! FOR EXAMPLE, A COMPANY HIRED TO ESTIMATE THE COMPOSITION OF A *GARBAGE DUMP* WAS FACED WITH SOME INTERESTING PROBLEMS NOT FOUND IN YOUR STANDARD TEXT...

Communication

BRILLIANT ANALYSIS IS WORTHLESS UNLESS THE RESULTS ARE CLEARLY COMMUNICATED IN PLAIN LANGUAGE, INCLUDING THE DEGREE OF STATISTICAL UNCERTAINTY IN THE CONCLUSIONS. FOR INSTANCE, THE MEDIA NOW MORE REGULARLY REPORT THE MARGIN OF ERRORS IN THEIR POLLING RESULTS.

Teamwork

IN OUR COMPLEX SOCIETY, THE SOLUTION TO MANY PROBLEMS REQUIRES A *TEAM EFFORT*. ENGINEERS, STATISTICIANS, AND ASSEMBLY LINE WORKERS ARE COOPERATING TO IMPROVE THE QUALITY OF THEIR PRODUCTS. BIOSTATISTICIANS, DOCTORS, AND AIDS ACTIVISTS ARE NOW WORKING TOGETHER TO DESIGN CLINICAL TRIALS TO MORE RAPIDLY EVALUATE THERAPIES

WELL, THAT'S IT! BY NOW, YOU SHOULD BE ABLE TO DO ANYTHING WITH STATISTICS, EXCEPT ***LIE, CHEAT, STEAL,*** AND ***GAMBLE.***


DO YOU HAVE ADEQUATE STATISTICAL MALPRACTICE INSURANCE?

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THESE TEXTS ARE CURRENT, CORRECT, LITERATE, AND WITTY. BESIDES THE ONES WE CITE, THERE ARE HUNDREDS OF TEXTBOOKS OUT THERE, AND WE WOULD RATE MOST AS AT LEAST ACCEPTABLE

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STATISTICAL SOFTWARE

IN THIS BOOK WE USED THE *MINITAB* STATISTICAL SOFTWARE SYSTEM (MINITAB INC., STATE COLLEGE, PA). THE PENN STATE STUDENT HEIGHT AND WEIGHT DATA IS FROM THE *PULSE* DATA SET ON THIS SYSTEM. COMPUTER GRAPHICS WERE GENERATED BY *S-PLUS* (STATISTICAL SCIENCES INC., SEATTLE WA), ON A 486 PC CLONE. S IS SOPHISTICATED SOFTWARE, DEVELOPED BY AT&T BELL LABS FOR ADVANCED ANALYSIS AND GRAPHICAL DISPLAYS.

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THERE ARE MANY HIGH QUALITY SOFTWARE PACKAGES AVAILABLE FOR THE PERSONAL COMPUTER, INCLUDING:

DATADESK (DATA DESCRIPTION, ITHACA, NY), FOR THE MACINTOSH

SAS (SAS INSTITUTE INC. CARY, NC), *SPSS* (SPSS INC, CHICAGO, IL), AND *BMDP* (BMDP STATISTICAL SOFTWARE, INC., LOS ANGELES, CA) WERE ORIGINALLY DESIGNED FOR MAINFRAME SYSTEMS AND NOW HAVE MIGRATED TO THE PC, COMPLETE WITH WINDOWS.

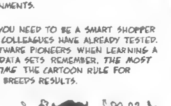
STATGRAPHICS (STATISTICAL GRAPHICS CORP. PRINCETON, NJ), FOR THE PC.

STATVIEW (ABACUS CONCEPTS, OAKLAND CA) FOR THE MACINTOSH

SYSTAT (SYSTAT, INC., EVANSTON IL) HAS SYSTEMS THAT RUN IN ALL ENVIRONMENTS.

THESE PACKAGES DIFFER IN IMPORTANT DETAILS. YOU NEED TO BE A SMART SHOPPER. WE RECOMMEND CHOOSING A SYSTEM THAT YOUR COLLEAGUES HAVE ALREADY TESTED. FEW OF US ARE CUT OUT TO BE STATISTICAL SOFTWARE PIONEERS. WHEN LEARNING A NEW SYSTEM, EXPERIMENT WITH SMALL, FAMILIAR DATA SETS. REMEMBER, *THE MOST EXPENSIVE PART OF ANY SOFTWARE IS YOUR TIME*. THE CARTOON RULE FOR LEARNING STATISTICAL COMPUTING IS: FAMILIARITY BREEDS RESULTS.

TRYING TO LEARN STATISTICAL *THEORY* AND STATISTICAL *COMPUTING* AT THE SAME TIME IS A LITTLE LIKE TRYING TO *WALK* AND *CHEW GUM* AT THE SAME TIME. DIFFERENT SKILLS AND THOUGHT PROCESSES ARE INVOLVED IN EACH. SET ASIDE SEPARATE TIMES TO LEARN THESE SUBJECTS, THEN BRING THEM TOGETHER. IN THIS WAY, YOU CAN BECOME A *CHEWING, WALKING, COMPUTING, RENAISSANCE STATISTICIAN!*



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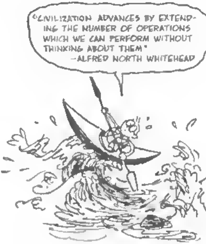

LARRY GONICK IS THE AUTHOR OR COAUTHOR OF A NUMBER OF BOOKS OF NON-FICTION CARTOONING, AS WELL AS THE FEATURE *SCIENCE CLASSICS*, APPEARING BIMONTHLY IN *DISCOVER* MAGAZINE. A DROP-OUT OF HARVARD GRADUATE SCHOOL IN MATHEMATICS, HE SEEMS TO HAVE COME FULL CIRCLE. HE LIVES IN SAN FRANCISCO WITH HIS WIFE, LISA, AND TWO DAUGHTERS, SOPHIE AND ANNA, AND HOPES TO UNDERSTAND LIFE WHILE REMAINING CHAINED TO HIS DRAWING BOARD.

If you have ever looked for P-values by shopping at P mart, tried to watch the Bernoulli Trials on "People's Court," or think that the standard deviation is a criminal offense in six states, then you need *The Cartoon Guide to Statistics* to put you on the road to statistical literacy.

The Cartoon Guide to Statistics covers all the central ideas of modern statistics: the summary and display of data, probability in gambling and medicine, random variables, Bernoulli Trials, the Central Limit Theorem, hypothesis testing, confidence interval estimation, and much more—all explained in simple, clear, and, yes, funny illustrations. Never again will you order the Poisson Distribution in a French restaurant!

LARRY GONICK is the author or coauthor of four previous cartoon guides by HarperCollins. WOOLLCOTT SMITH is a statistician active in research and consulting, and a professor at Temple University.

"Gonick is one of a kind." —*Discover*

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